Section 1.4 Probability Space

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1. Axiomatic Definition of Probability
2. Properties of Probability
3. Exercises

Axiomatic Definition of Probability

Definition

A collection $\mathscr{F}$ of subsets of Ω is a ***σ-algebra*** *if*
(i) $\Omega \in \mathscr{F}$;
(ii) $F \in \mathscr{F} \Longrightarrow \overline{F} \in \mathscr{F}$;
(iii) *if F_n is a countable collection of sets, $n = 1, 2, \cdots$ such that $F_n \in \mathscr{F}$ for all n, then*

$$\bigcup_n F_n \in \mathscr{F}.$$

What is the intuition behind a σ-algebra?

If Ω represents the collection of possible outcomes of an experiment, a subset of Ω is called an event. Then, a σ-algebra represents the collection of all possible, interesting events from the viewpoint of a given experiment.

Set of Sets

Example

Let the experiment be a coin toss (where we blow on the coin if it stands up straight!). Then $\Omega = \{H, T\}$. Let $\mathscr{F}_1 = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}$ and $\mathscr{F}_2 = \{\emptyset, \{H, T\}\}$. It is easy to prove that both $\mathscr{F}_1$ and $\mathscr{F}_2$ are σ-algebra on Ω.

Now we know that the reason that Bertrand Paradox occurs is σ-algebra on Ω is not uniquely determined.

σ-Algebra: domain of probability

(i) $\Omega \in \mathscr{F}$

(ii) $A \in \mathscr{F} \Rightarrow \bar{A} \in \mathscr{F}$ $\Bigg\} \Rightarrow \emptyset \in \mathscr{F}$

(iii) $A_i \in \mathscr{F}$, $i = 1, 2, \ldots \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathscr{F} \Rightarrow \bigcap_{i=1}^{\infty} A_i \in \mathscr{F}$

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(De-Morgan)

Axiomatic Definition of Probability

Definition

*Let $P(A)(A \in \mathscr{F})$ be a non-negative set function on the σ-algebra $\mathscr{F}$. $P(A)$ is called the probability measure or **probability of event** A if it satisfies the following three axioms:*

Axiom 1. *for every $A \in \mathscr{F}, P(A) \geqslant 0$;*

Axiom 2. $P(\Omega) = 1$;

Axiom 3. *(countable additivity) for every infinite sequence of countable disjoint events $A_1, A_2, \cdots$,*

$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$

*The sets in σ-algebra $\mathscr{F}$ are called events. $\mathscr{F}$ is called to be the algebra of events. The triple $(\Omega, \mathscr{F}, P)$ is a **probability space** or **probability triple**.*

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Properties of Probability

Theorem

$P(\emptyset) = 0$.

Proof. Consider the infinite sequence of events $A_1, A_2, \cdots$ such that $A_i = \emptyset$ for $i = 1, 2, \cdots$. In other words, each of the events in the sequence is just the empty set $\emptyset$. This sequence is a sequence of disjoint events, since $\emptyset \cap \emptyset = \emptyset$. Furthermore, $\cup_{i=1}^{\infty} A_i = \emptyset$. Therefore, it follows from Axiom 3 that

$$P(\emptyset) = P\big(\bigcup_{i=1}^{\infty} A_i\big) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} P(\emptyset).$$

This equation states that when the number $P(\emptyset)$ is added repeated in an infinite series, the sum of that series is simply the number $P(\emptyset)$. The only real number with this property ie zero.

Properties of Probability

Theorem

For every finite sequence of countable disjoint events $A_1, A_2, \cdots, A_n$,

$$P\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{i=1}^{n} P(A_i).$$

Proof. Consider the infinite sequence of events $A_1, A_2, \cdots$, in which $A_1, A_2, \cdots, A_n$ are the n given disjoint events and $A_i = \emptyset$ for $i > n$. Then the events in this infinite sequence are disjoint and $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{n} A_i$. Therefore,

$$\begin{aligned} P\Big(\bigcup_{i=1}^{n} A_i\Big) =& P\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{n} P(A_i) + \sum_{i=n+1}^{\infty} P(A_i) \\ &= \sum_{i=1}^{n} P(A_i) + 0 = \sum_{i=1}^{n} P(A_i) \end{aligned}$$

holds by Axiom 3. $\square$

Properties of Probability

Theorem

For every event A, $P(\overline{A}) = 1 - P(A)$.

Proof. Since A and $\overline{A}$ are disjoint events and $A \cup \overline{A} = \Omega$, $P(\Omega) = P(A) + P(\overline{A})$. By Axiom 2, we know $P(\Omega) = 1$, then $P(\overline{A}) = 1 - P(A)$. $\square$

Theorem

If $A \subset B$, then $P(B - A) = P(B) - P(A)$ and $P(A) \leqslant P(B)$.

Proof. Since $A \subset B$, the event B can be treated as the union of the two disjoint events A and $B\overline{A}$. Therefore, $P(B) = P(A \cup B\overline{A}) = P(A) + P(B\overline{A})$, i.e.,

$$P(B - A) = P(B\overline{A}) = P(B) - P(A).$$

Since $P(B - A) \geqslant 0$, $P(B) \geqslant P(A)$. $\square$

Properties of Probability

Theorem

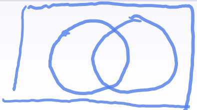
For every event A, $0 \leqslant P(A) \leqslant 1$.

Proof. It is known from Axiom 1 that $P(A) \geqslant 0$. Since $A \subset \Omega$ for every event A, Theorem 1.3.4 implies $P(A) \leqslant P(\Omega) = 1$, by Axiom 2. □

Theorem

For every two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(AB). \tag{1}$$



Properties of Probability

Proof. The event $A \cup B$ may be treated as the union of the three events $A\overline{B}, AB$ and $\overline{A}B$. Since the three events are disjoint,

$$P(A \cup B) = P(A\overline{B} \cup AB \cup \overline{A}B) = P(A\overline{B}) + P(AB) + P(\overline{A}B). \quad (2)$$

Furthermore, it can be seen that

$$P(A\overline{B}) = P(A - AB) = P(A) - P(AB)$$

and

$$P(\overline{A}B) = P(B - AB) = P(B) - P(AB).$$

Then

$$P(A \cup B) = P(A) + P(B) - P(AB).$$

$\square$

Properties of Probability

Corralary

For every three events A_1, A_2 *and* A_3,

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) =& P(A_1) + P(A_2) + P(A_3) \\ & - [P(A_1A_2) + P(A_2A_3) + P(A_1A_3)] \\ & + P(A_1A_2A_3). \end{aligned} \tag{3}$$

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Corralary

For every n *events* $A_1, A_2, \cdots, A_n$,

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$$\begin{aligned} P(\bigcup_{i=1}^{n} A_i) =& \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_iA_j) + \sum_{i<j<k} P(A_iA_jA_k) \\ & - \sum_{i<j<k<l} P(A_iA_jA_kA_l) + \cdots + (-1)^{n+1} P(A_1A_2\cdots A_n). \end{aligned} \tag{4}$$

Properties of Probability

Theorem

(i) *Let $A_1, A_2, \cdots$ be an infinite sequence of events such that $A_1 \subset A_2 \subset \cdots$. Then*

$$P\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \lim_{n\to\infty} P(A_n). \tag{5}$$

(ii) *Let $A_1, A_2, \cdots$ be an infinite sequence of events such that $A_1 \supset A_2 \supset \cdots$. Then*

$$P\Big(\bigcap_{i=1}^{\infty} A_i\Big) = \lim_{n\to\infty} P(A_n). \tag{6}$$

Properties of Probability

Proof. (i) Let $A = \bigcup_{i=1}^{\infty} A_i$. Since $A_n \subset A_{n+1}$, we rewrite A_n by n disjoint events $A_1, A_2\overline{A_1}, \cdots, A_n\overline{A_{n-1}}$:

$$A_n = A_1 \cup A_2\overline{A_1} \cup \cdots \cup A_n\overline{A_{n-1}}.$$

Let $B_n = A_n\overline{A_{n-1}}$, $n = 1, 2, \cdots$. Then

$$P(A) = P\Big(\bigcup_{m=1}^{\infty} B_m\Big) = \sum_{m=1}^{\infty} P(B_m) = \lim_{n\to\infty} \sum_{m=1}^{n} P(B_m).$$

Thus $\sum_{m=1}^{n} P(B_m) = P\Big(\bigcup_{m=1}^{n} B_m\Big) = P(A_n)$. So

$P\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \lim_{n\to\infty} P(A_n)$.

(ii) It can be obtained by defining $B_i = \overline{A_i}$. $\square$

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Exercises

Example

Assume A and B are two events such that $P(A) = 1/3$ and $P(B) = 1/2$. Determine the value of $P(B-A)$ satisfying each of the following conditions. (a) A and B are disjoint; (b) $A \subset B$; (c) $P(AB) = 1/8$.

Solution. Since

$$P(B) = P(AB) + P(\overline{A}B) = P(AB) + P(B-A),$$

That is,

$$P(B-A) = P(B) - P(AB). \tag{7}$$

(a) Since A and B are disjoint means $AB = \emptyset$, we know

$$P(B-A) = P(B) - P(AB) = P(B) = 1/2.$$

(b) If $A \subset B$, then $AB = A$. Thus
$P(B-A) = P(B) - P(AB) = P(B) - P(A) = 1/6.$
(c) $P(B-A) = P(B) - P(AB) = 1/2 - 1/8 = 3/8.$ $\square$

Example

Assume A and B are two events such that $P(B) = b$ and $P(A \cup B) = c$. $(c \geq b)$ Determine the value of $P(A\overline{B})$.

Solution. By equations (9) and (7),

$$\begin{aligned} P(A \cup B) =& P(A) + P(B) - P(AB) \\ =& P(B) + [P(A) - P(AB)] \\ =& P(B) + P(A - B). \end{aligned}$$

Thus $P(A\overline{B}) = P(A - B) = P(A \cup B) - P(B) = c - b$.

$\square$

Exercises

Example

Select an integer from 1 to 1000 at random. Calculate the probability of the event that the integer is not divisible by 6 and 8.

Solution. Let A be the event that the integer is divisible by 6, and B be the event that the integer is divisible by 8. The required probability is

$$\begin{aligned}P(\overline{A}\ \overline{B}) =& P(\overline{A \cup B}) = 1 - P(A \cup B) \\ =& 1 - [P(A) + P(B) - P(AB)] \\ =& 1 - 166/1000 - 125/1000 + 41/1000 \approx 0.75.\end{aligned}$$

$\square$

Exercises

Example

(**The matching problem**) An absent-minded secretary prepares n letters and envelopes to send to n different people, but then randomly students the letters into the envelopes. A match occurs if a letter is inserted in the proper envelope. Here we shall determine the probability of the event that at least one match occurs.

Exercises

Solution. Let A_i be the event that letter i is placed in the correct envelop ($i = 1, 2, \cdots, n$) (or a person gets his or her own hat) and we shall determine the value of $P(\bigcup_{i=1}^{n} A_i)$ by using equation (4). Since

$$P(A_i) = \frac{(n-1)!}{n!} = \frac{1}{n}, \qquad i = 1, 2, \cdots ;$$

$$P(A_i A_j) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}, \qquad 1 \leqslant i < j \leqslant n;$$

$$P(A_i A_j A_k) = \frac{(n-3)!}{n!} = \frac{1}{n(n-1)(n-2)}, \qquad 1 \leqslant i < j < k \leqslant n;$$

$$\vdots$$

$$P(A_1 \cdots A_n) = \frac{1}{n!},$$

□

Solution. we have

$$P\Big(\bigcup_{i=1}^{n} A_i\Big) = \binom{n}{1}\cdot\frac{1}{n} - \binom{n}{2}\cdot\frac{1}{n(n-1)} + \cdots + (-1)^{n+1}\frac{1}{n!}$$
$$= 1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + (-1)^{n+1}\frac{1}{n!}.$$

$\square$

Exercises

This probability has the following interesting features. As $n \to \infty$, the value of $P(\bigcup_{i=1}^{n} A_i)$ approaches the following limit:

$$1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots .$$

It is shown in books on elementary calculus that the sum of the infinite series on the right side of this equation is $1 - (1/e)$, where $e = 2.71828\cdots$. Hence, $1 - (1 - 1/e) = 0.63212\cdots$. It follows that for large value of n, the probability that at least one letter will be placed in the correct envelope is approximately 0.63212.

Exercises

The value of $P(\bigcup_{i=1}^{n} A_i)$ converges to the limit very rapidly. In fact, for $n = 7$ the exact value $P(\bigcup_{i=1}^{7} A_i)$ and the limiting value 0.63212 agree to four decimal places. Hence, regardless of whether seven letters are placed at random in seven or seven million letters are placed at random in seven million envelopes, the probability that at least one letter will be replaced in the correct envelope is 0.63212.

Exercises

- **Definition**
  - **Axiom 1**. for every $A \in \mathscr{F}, P(A) \geqslant 0$;
  - **Axiom 2**. $P(\Omega) = 1$;
  - **Axiom 3**. (countable additivity)
- **Properties**
  - $P(\emptyset) = 0$.
  - For disjoint events $A_1, A_2, \cdots, A_n$,

$$P\big(\bigcup_{i=1}^{n} A_i\big) = \sum_{i=1}^{n} P(A_i).$$

  - For every event A, $P(\overline{A}) = 1 - P(A)$.
  - If $A \subset B$, then $P(B - A) = P(B) - P(A)$ and $P(A) \leqslant P(B)$.
  - For every event A, $0 \leqslant P(A) \leqslant 1$.
  - For every two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(AB). \tag{9}$$

Thank you for your patience!