Section 3.3 Conditional Distributions

School of Sciences, BUPT

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Discrete Case

If X and Y are discrete random variables, then it is natural to define the ***conditional probability function of*** X ***given that*** $Y = y$, by

p.f.

可以此求 $p_{X|Y}(x|y)$

$$p_{X|Y}(x|y) = P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{p(x, y)}{p_Y(y)}$$

for all values of y such that $p_Y(y) > 0$. Similarly, the ***conditional probability function of*** Y ***given that*** $X = x$ is given by

$$p_{Y|X}(y|x) = P(Y = y|X = x) = \frac{P(X = x, Y = y)}{P(X = x)} = \frac{p(x, y)}{p_X(x)}$$

for all values of x such that $p_X(x) > 0$.

Discrete Case

Example

Suppose that $p(x, y)$, the joint probability function of X and Y, is given by $p(0, 0) = 0.4$, $p(0, 1) = 0.2$, $p(1, 0) = 0.1$, $p(1, 1) = 0.3$. Calculate the conditional probability function of X given that $Y = 1$.

Solution. We first note that

$$p_Y(1) = \sum_x p(x, 1) = p(0, 1) + p(1, 1) = 0.5.$$

Hence,

$$p_{X|Y}(0|1) = \frac{p(0, 1)}{p_Y(1)} = \frac{2}{5}$$

and

$$p_{X|Y}(1|1) = \frac{p(1, 1)}{p_Y(1)} = \frac{3}{5}.$$

□

Discrete Case

Example

Suppose that X is a random variable, which can be selected randomly from the numbers 1, 2, 3 and 4. Let Y be an another random variable, which can be selected randomly from positive integers 1 to X. Find the joint probability function of (X, Y).

Solution. It is obvious that all possible values of X and Y are 1, 2, 3 and 4. So we have 4×4 different pairs of (X, Y). Then

$$P(X = 1, Y = 1) = P(X = 1) \cdot P(Y = 1|X = 1) = \frac{1}{4} \cdot 1 = \frac{1}{4}.$$

$$P(X = 2, Y = 1) = P(X = 2) \cdot P(Y = 1|X = 2) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}.$$

$$P(X = 3, Y = 1) = P(X = 3) \cdot P(Y = 1|X = 3) = \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}.$$

$$P(X = 4, Y = 1) = P(X = 4) \cdot P(Y = 1|X = 4) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}.$$

Discrete Case

y \ x	1	2	3	4
1	$1/4$	$1/4 \cdot 1/2$	$1/4 \cdot 1/3$	$1/4 \cdot 1/4$
2	0	$1/4 \cdot 1/2$	$1/4 \cdot 1/3$	$1/4 \cdot 1/4$
3	0	0	$1/4 \cdot 1/3$	$1/4 \cdot 1/4$
4	0	0	0	$1/4 \cdot 1/4$

□

Discrete Case

Example

A store has a certain goods. Let X be the number of customers entering this store in a specified period of time. Suppose that $X \sim P(\lambda)$ and the probability of the event that each customer purchases the certain goods is p. If customs are independent, then find the probability function of the number of customers who purchase the certain goods.

Solution. Let Y be the number of customers who purchase the certain goods. Since $X \sim P(\lambda)$,

$$P(X = m) = \frac{\lambda^m}{m!} e^{-\lambda}, \quad m = 0, 1, 2, \cdots .$$

Discrete Case

Given that $X = m$, i.e., the number of customers entering this store is m, the conditional probability function of Y is binomial distribution, i.e.,

$$P(Y = k|X = m) = \binom{m}{k} p^k (1-p)^{m-k}, \quad k = 0, 1, 2, \cdots, m.$$

Using total probability formula, we have

$$\begin{aligned} P(Y = k) &= \sum_{m=k}^{+\infty} P(X = m) P(Y = k|X = m) \\ &= \sum_{m=k}^{+\infty} \frac{\lambda^m}{m!} e^{-\lambda} \binom{m}{k} p^k (1-p)^{m-k} \\ &= e^{-\lambda} \frac{\lambda^k}{k!} \sum_{m=k}^{+\infty} \frac{[(1-p)\lambda]^{m-k}}{(m-k)!} \\ &= e^{-\lambda} \frac{\lambda^k}{k!} e^{\lambda(1-p)} = \frac{\lambda^k}{k!} e^{-\lambda p}, \quad k = 0, 1, 2, \cdots \end{aligned}$$

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Continuous Case

Definition

If X and Y have joint p.d.f. $f(x,y)$, then ***the conditional probability density function of*** X ***given*** $Y = y$ *is given by*

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{f(x,y)}{f_Y(y)} & if \ 0 < f_Y(y) < +\infty, \\ 0 & elsewhere. \end{cases} \tag{1}$$

Similarly, the ***conditional probability density function of*** Y ***given*** $X = x$ *is given by*

$$f_{Y|X}(y|x) = \begin{cases} \dfrac{f(x,y)}{f_X(x)} & if \ 0 < f_X(x) < +\infty, \\ 0 & elsewhere. \end{cases} \tag{2}$$

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In fact, for each fixed value of y such that $f_Y(y) > 0$, the function $f_{X|Y}(x|y)$ will be a p.d.f. for X over the real line since $f_{X|Y}(x|y) \geqslant 0$ and $\int_{-\infty}^{+\infty} f_{X|Y}(x|y)dx = 1$. Similarly, for each fixed value of x such that $f_X(x) > 0$, the function $f_{Y|X}(y|x)$ is also a p.d.f. for Y over the real line.

Continuous Case

Remark

A conditional p.d.f. is not the result of conditional on a set of probability zero. Actually, the value of $f_{X|Y}(x|y)$ *is a limit:*

$$f_{X|Y}(x|y) = \lim_{\varepsilon \to 0} \frac{\partial}{\partial x} P(X \leqslant x | y - \varepsilon < Y \leqslant y + \varepsilon). \qquad (3)$$

The conditioning event $\{y - \varepsilon < Y \leqslant y + \varepsilon\}$ *in equation (3) has positive probability if the marginal p.d.f. of* Y *is positive at* y.

Continuous Case

Example

Given

$$f(x,y) = \begin{cases} k & \text{for } 0 < x < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Determine $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

Solution. The joint p.d.f. is given to be a constant in the region $\{(x,y) : 0 < x < y < 1\}$. This gives

$$\iint_{\mathbb{R}^2} f(x,y)dxdy = \int_0^1 \int_0^y kdxdy = \int_0^1 kydy = \frac{k}{2} = 1 \Rightarrow k = 2.$$

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)dy = \int_x^1 2dy = 2(1-x), \qquad 0 < x < 1,$$

Continuous Case

and

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)dx = \int_0^y 2dx = 2y, \qquad 0 < y < 1.$$

From equations (1) and (2), we get if $0 < y < 1$, then

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{y} & \text{for } 0 < x < y, \\ 0 & \text{otherwise.} \end{cases}$$

and if $0 < x < 1$, then

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \begin{cases} \dfrac{1}{1-x} & \text{for } x < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

□

Continuous Case

Example

Given

$$f(x,y) = \begin{cases} \dfrac{3}{16}(4-2x-y) & \text{for } x>0,\ y>0,\ 2x+y<4, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine $f_{Y|X}(y|x)$.
(b) Determine the value of $P(Y \geqslant 2 \mid X \leqslant 1/2)$.
(c) Determine the value of $P(Y \geqslant 2 \mid X = 1/2)$.

Continuous Case

Solution. (a) First let us find the marginal p.d.f. $f_X(x)$.
If $0 < x < 2$, then

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)dy = \int_0^{4-2x} \frac{3}{16}(4-2x-y)dy = \frac{3}{8}(2-x)^2.$$

If $x \leqslant 0$ or $x \geqslant 1$, then $f(x,y) = 0$. i.e.,

$$f_X(x) = \begin{cases} \frac{3}{8}(2-x)^2 & \text{for } 0 < x < 2, \\ 0 & \text{otherwise,} \end{cases}$$

So when $0 < x < 2$, we have

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \begin{cases} \frac{4-2x-y}{2(2-x)^2} & \text{for } 0 < y < 4-2x, \\ 0 & \text{otherwise.} \end{cases}$$

Continuous Case

(b) According to the definition of conditional probability, we have

$$\begin{aligned} P(Y \geqslant 2 \mid X \leqslant 1/2) &= \frac{P(X \leqslant 1/2, Y \geqslant 2)}{P(X \leqslant 1/2)} \\ &= \frac{\displaystyle\int_0^{1/2} dx \int_2^{4-2x} \frac{3}{16}(4 - 2x - y)dy}{\displaystyle\int_0^{1/2} \frac{3}{8}(2-x)^2 dx} \\ &= \frac{\displaystyle\int_0^{1/2} (\frac{3}{8} - \frac{3}{4}x + \frac{3}{8}x^2)dx}{\displaystyle\int_0^{1/2} \frac{3}{8}(2-x)^2 dx} = \frac{7}{64}. \end{aligned}$$

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(c) Since

$$f_{Y|X}\left(y\middle|\frac{1}{2}\right) = f_{Y|X}(y|x)\Big|_{x=1/2} = \begin{cases} \dfrac{2(3-y)}{9} & \text{for } 0 < y < 3, \\ 0 & \text{otherwise,} \end{cases}$$

we get

$$\begin{aligned} P(Y \geqslant 2 \mid X = 1/2) &= \int_2^{+\infty} f_{Y|X}\left(y\middle|\frac{1}{2}\right) dy \\ &= \int_2^3 \frac{2(3-y)}{9} dy = \frac{1}{9}. \end{aligned}$$

□

Continuous Case

Example

Suppose that $(X,Y) \sim N(\mu_1, \mu_2; \sigma_1^2, \sigma_2^2; \rho)$. Find $f_{Y|X}(y|x)$ and $f_{X|Y}(x|y)$.

Solution. Given $X = x$, the conditional p.d.f. of Y is

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{1}{\sigma_2\sqrt{2\pi(1-\rho^2)}} \exp\left\{ -\frac{1}{2(1-\rho^2)\sigma_2^2}\left(y - \mu_2 - \rho\sigma_2\frac{x-\mu_1}{\sigma_1}\right)\right.$$

This is also a normal distribution

$$N\left(\mu_2 + \rho\sigma_2\frac{x-\mu_1}{\sigma_1}, (1-\rho^2)\sigma_2^2\right).$$

Here x is a constant. Similarly, given $Y = y$, the conditional p.d.f. of X is a normal distribution
$N\left(\mu_1 + \rho\sigma_1\frac{y-\mu_2}{\sigma_2}, (1-\rho^2)\sigma_1^2\right)$. □

Continuous Case

For two dimension normal distribution, we can see not only the marginal distributions are normal, but the conditional distributions are also normal.

Summary

- **Discrete**

$$p_{X|Y}(x|y) = P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{p(x, y)}{p_Y((y)}$$

$$p_{Y|X}(y|x) = P(Y = y|X = x) = \frac{P(X = x, Y = y)}{P(X = x)} = \frac{p(x, y)}{p_X(x)}$$

- **Continuous**

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{f(x, y)}{f_Y(y)} & if \;\; 0 < f_Y(y) < +\infty, \\ 0 & elsewhere. \end{cases}$$

$$f_{Y|X}(y|x) = \begin{cases} \dfrac{f(x, y)}{f_X(x)} & if \;\; 0 < f_X(x) < +\infty, \\ 0 & elsewhere. \end{cases}$$

Thank you for your patience !