Section 3.4 One Function of Two Random Variables

School of Sciences, BUPT

One Function of Two Random Variables

Given two random variables X and Y and a function $\varphi(x, y)$, we form a new random variable Z as

$$Z = \varphi(X, Y).$$

Given the joint p.d.f. $f(x, y)$ of X and Y, how does one obtain the p.d.f. $f_Z(z)$ of Z?

Contents

Discrete Case

Example

Assume the joint p.f. of random variable (X, Y) is given by Table

$y \backslash x$	0	1	2	3	4	5
0	0	0.01	0.03	0.05	0.07	0.09
1	0.01	0.02	0.04	0.05	0.06	0.08
2	0.01	0.03	0.05	0.05	0.05	0.06
3	0.01	0.02	0.04	0.06	0.06	0.05

Determine the probability functions of (a) $X + Y$, (b) $\min(X, Y)$ and (c) $\max(X, Y)$.

Discrete Case

Solution. (a) The possible outcomes of $X+Y$ are: $0, 1, 2, 3, 4, 5, 6, 7, 8$. Then

$$
\begin{aligned}
P(X+Y=0) &= P(X=0, Y=0) = 0, \\
P(X+Y=1) &= P(\{X=0, Y=1\} \cup \{X=1, Y=0\}) \\
&= P(X=0, Y=1) + P(X=1, Y=0) \\
&= 0.01 + 0.01 = 0.02, \\
P(X+Y=2) &= P(\{X=0, Y=2\} \cup \{X=1, Y=1\} \cup \{X=2, Y= \\
&= P(X=0, Y=2) + P(X=1, Y=1) + P(X=2, Y \\
&= 0.03 + 0.02. + 0.01 = 0.06.
\end{aligned}
$$

Similarly, we can calculate other probabilities.

$X+Y$	0	1	2	3	4	5	6	7	8
P	0	0.02	0.06	0.13	0.19	0.24	0.19	0.12	0.05

Discrete Case

(b) The possible outcomes of $\min(X, Y)$ are: $0, 1, 2, 3$. Then

$$
\begin{aligned}
& P(\min(X,Y) = 0) \\
= {} & P\Big(\{X = 0, Y = 3\} \cup \{X = 0, Y = 2\} \cup \{X = 0, Y = 1\} \\
& \cup \{X = 0, Y = 0\} \cup \{X = 1, Y = 0\} \cup \{X = 2, Y = 0\} \\
& \cup \{X = 3, Y = 0\} \cup \{X = 4, Y = 0\} \cup \{X = 5, Y = 0\}\Big) \\
= {} & P(X = 0, Y = 3) + P(X = 0, Y = 2) + P(X = 0, Y = 1) \\
& + P(X = 0, Y = 0) + P(X = 1, Y = 0) + P(X = 2, Y = 0) \\
& + P(X = 3, Y = 0) + P(X = 4, Y = 0) + P(X = 5, Y = 0) \\
= {} & 0.01 + 0.01 + 0.01 + 0 + 0.01 + 0.03 + 0.05 + 0.07 + 0.09 = 0.28.
\end{aligned}
$$

Other probabilities can be calculated similarly.

$\min(X, Y)$	0	1	2	3
P	0.28	0.30	0.25	0.17

Discrete Case

(c) The possible outcomes of $\max(X, Y)$ are: $0, 1, 2, 3, 4, 5$. Then

$$P(\max(X, Y) = 0) = P(\{X = 0, Y = 0\}) = 0.$$

$$\begin{aligned}
&P(\max(X, Y) = 1) \\
=\ &P(\{X = 1, Y = 0\} \cup \{X = 0, Y = 1\} \cup \{X = 1, Y = 1\}) \\
=\ &P(X = 1, Y = 0) + P(X = 1, Y = 1) + P(X = 1, Y = 0) \\
=\ &0.01 + 0.02 + 0.01 = 0.04.
\end{aligned}$$

Other probabilities can be calculated similarly.

$\max(X, Y)$	0	1	2	3	4	5
P	0	0.04	0.16	0.28	0.24	0.28

□

Discrete Case

It is often important to be able to calculate the distribution of $X + Y$ from the distributions of X and Y when X and Y are independent.

Theorem

If X and Y are independent discrete random variables, then $X + Y$ has probability function

$$p_{X+Y}(n) = \sum_k p_X(k) p_Y(n-k). \tag{1}$$

The function p_{X+Y} is called the convolution of p_X and p_Y, and is written as

$$p_{X+Y} = p_X * p_Y.$$

Discrete Case

Example

Assume X and Y are independent, and $X \sim B(n_1, p)$, $Y \sim B(n_2, p)$. Prove $X + Y \sim B(n_1 + n_2, p)$.

Proof.

$$\begin{aligned}
P(X+Y=k) &= \sum_{k_1} P(X=k_1)P(Y=k-k_1) \\
&= \sum_{k_1=0}^{k} \binom{n_1}{k_1} p^{k_1}(1-p)^{n_1-k_1} \cdot \binom{n_2}{k-k_1} p^{k-k_1}(1-p)^{n_2-k+k_1} \\
&= p^k(1-p)^{n_1+n_2-k} \sum_{k_1=0}^{k} \binom{n_1}{k_1} \cdot \binom{n_2}{k-k_1} \\
&= \binom{n_1+n_2}{k} p^k (1-p)^{n_1+n_2-k}, \qquad k = 0, 1, 2, \cdots, n_1+n_2.
\end{aligned}$$

Thus $X + Y \sim B(n_1 + n_2, p)$. □

Discrete Case

Example

Assume X and Y are independent, and $X \sim P(\lambda)$, $Y \sim P(\mu)$. Prove $X+Y \sim P(\lambda+\mu)$.

Solution. $X+Y$ has the following probability function:

$$\begin{aligned}
P(X+Y=n) &= \sum_{k=0}^{n} P(X=k, Y=n-k) = \sum_{k=0}^{n} P(X=k)P(Y=n-k) \\
&= \sum_{k=0}^{n} e^{-\lambda}\frac{\lambda^k}{k!} e^{-\mu}\frac{\mu^{n-k}}{(n-k)!} = e^{-(\lambda+\mu)}\sum_{k=0}^{n}\frac{\lambda^k \mu^{n-k}}{k!(n-k)!} \\
&= \frac{e^{-(\lambda+\mu)}}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\lambda^k\mu^{n-k} \\
&= \frac{e^{-(\lambda+\mu)}}{n!}(\lambda+\mu)^n
\end{aligned}$$

Thus, $X+Y$ has a Poisson distribution with parameter $\lambda+\mu$. □

Distribution Functions of Multiple Random Vectors

Proposition

(i) *If* $X_1,\ X_2,\cdots,\ X_m$ *are independent and* X_i *has a Bernoulli distribution with parameter* p*, for* $i=1,2,\ \cdots,n$*, then we have:*

$$\sum_{i=1}^{n} X_i \sim B(n,\ p), \quad for\ n=1,2,\cdots$$

More generally, if $X_1,\ X_2,\cdots,\ X_m$ *are independent, and* $X_i \sim B(n_i,p)$*,* $i=1,2,\cdots,m$*. Then* $X_1+X_2+\cdots+X_m \sim B(n_1+n_2+\cdots+n_m,\ p)$*.*

(ii) *Assume* $X_1,\ X_2,\cdots,\ X_n$ *are independent, and* $X_i \sim P(\lambda_i)$*,* $i=1,2,\cdots,n$*. Then* $X_1+X_2+\cdots+X_n \sim P(\lambda_1+\cdots+\lambda_n)$*.*

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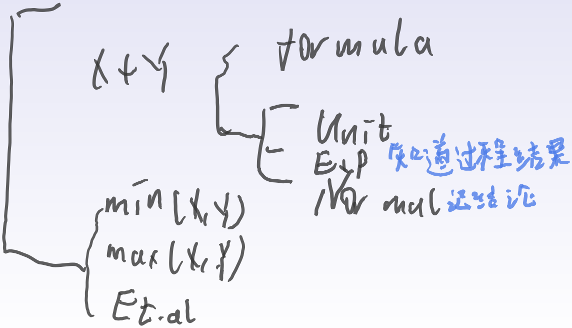

Continuous Case

Let X and Y be random variables having joint p.d.f. $f(x, y)$. Let Z be given by $Z = \varphi(X, Y)$, where φ is a real-valued function whose domain contains the range of X and Y.

In order to determine the p.d.f. of Z, we need to find the d.f. of Z first. Thus

$$\begin{aligned} F_Z(z) =& P(Z \leqslant z) = P(\varphi(X, Y) \leqslant z) \\ =& P((X, Y) \in A_z) = \iint_{A_z} f(x, y) dx dy, \end{aligned}$$

where

$$A_z = \{(x, y) | \varphi(x, y) \leqslant z\}.$$

Thus $f_Z(z) = F'(z)$.

Continuous Case

In this section, we mainly concern the cases when $\varphi(X,Y) = X + Y$, $\min(X,Y)$ and $\max(X,Y)$.

1. The case of $\boldsymbol{X+Y}$

Set $Z = X + Y$. Then

$$A_z = \{(x,y) | x + y \leqslant z\}$$

is just the half-plane to the lower left of the line $x + y = z$. Thus

$$F_Z(z) = \iint\limits_{A_z} f(x,y)dxdy = \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{z-x} f(x,y)dy \Big) dx.$$

Make the change of variable $y = v - x$ in the inner integral. Then

Continuous Case

$$F_Z(z) = \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{z} f(x, v-x)dv \Big) dx$$
$$= \int_{-\infty}^{z} \Big(\int_{-\infty}^{\infty} f(x, v-x)dx \Big) dv,$$

where we have interchanged the order of integration. Thus the density of $Z = X + Y$ is given by

$$f_Z(z) = f_{X+Y}(z) = F'(z) = \int_{-\infty}^{\infty} f(x, z-x)dx, \qquad -\infty < z < \infty. \tag{2}$$

If X and Y are independent, then equation (2) can be rewritten as

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx, \qquad -\infty < z < \infty. \tag{3}$$

Continuous Case

That means the density of the sum of two independent random variables is the convolution of the individual densities.
Equation (3) can be written as

$$f_{X+Y} = f_X * f_Y.$$

Continuous Case

Example

Let X and Y be two independent random variables, uniformly distributed in the same interval $[0,1]$. Compute the distribution of $X+Y$, and compare with the distribution of $2X$.

Solution. The density of X is

$$f_X(x) = \begin{cases} 1 & \text{if } x \in (0,1), \\ 0 & \text{otherwise.} \end{cases}$$

The density of Y is the same. Thus $f_{X+Y}(z) = 0$ for $z \leqslant 0$. For $z > 0$,

$$f_X(x) f_Y(z-x) = \begin{cases} 1 & \text{if } 0 \leqslant x \leqslant 1,\ 0 \leqslant z-x \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

Continuous Case

If $0 \leqslant z \leqslant 1$, then

$$f_{X+Y}(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x)dx = \int_0^z 1dx = z.$$

If $1 < z \leqslant 2$, then

$$f_{X+Y}(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x)dx = \int_{z-1}^1 1dx = 2 - z.$$

If $2 < z < \infty$, then

$$f_{X+Y}(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x)dx = \int_0^z 0dx = 0.$$

Continuous Case

In summary,

$$f_{X+Y}(z) = \begin{cases} z & \text{if } 0 \leqslant z \leqslant 1, \\ 2 - z & \text{if } 1 < z \leqslant 2, \\ 0 & \text{elsewhere.} \end{cases}$$

But, obviously,

$$f_{2X}(z) = \begin{cases} 1/2 & \text{if } z \in (0, 2), \\ 0 & \text{otherwise.} \end{cases}$$

It is because X and Y are independent, but X and X are not. □

Continuous Case

Example

Let X and Y be independent random variables, which have exponential distributions with parameter λ_1 and λ_2 respectively. Find the distribution of $X + Y$.

Solution. Let the distributions of X and Y be respectively

$$f_X(x) = \begin{cases} \lambda_1 e^{-\lambda_1 x} & \text{for } x \geqslant 0, \\ 0 & \text{for } x < 0, \end{cases} \text{ and } f_Y(y) = \begin{cases} \lambda_2 e^{-\lambda_2 y} & \text{for } y \geqslant 0, \\ 0 & \text{for } y < 0. \end{cases}$$

For $z \leqslant 0$, we have $f_{X+Y}(z) = 0$. For $z > 0$,

$$\begin{aligned} f_{X+Y}(z) &= \int_0^z \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 (z-x)} dx \\ &= \lambda_1 \lambda_2 e^{-\lambda_2 z} \int_0^z e^{(-\lambda_1 + \lambda_2)x} dx \\ &= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} (e^{-\lambda_1 z} - e^{-\lambda_2 z}). \end{aligned}$$

□

Continuous Case

Example

Let X and Y be independent random variables having the respective normal densities $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$. Prove $X + Y$ has the normal distribution $N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Normal

$X - Y \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$

Corralary

If $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim (\mu_2, \sigma_2^2)$ *and* $X \perp Y$, *then*

$$aX + bY + c \sim N(a\mu_1 + b\mu_2 + c, a^2\sigma_1^2 + b^2\sigma_2^2),$$

where a, b *are constants.*

Continuous Case

Corralary

Assume X_1, $X_2, \cdots$, X_n *are independent, and* $X_i \sim N(\mu_i, \sigma_i^2)$, *for* $i = 1, \cdots, n$, *then*

$$\sum_{i=1}^{n} a_i X_i \sim N\Big(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\Big).$$

Continuous Case

By using similar ways of finding the p.d.f. of $X+Y$, we can obtain the probability density functions of $X-Y$, $X \cdot Y$ and X/Y.

(1) Let $Z = X - Y$. Then the p.d.f. of Z is

$$f_Z(z) = \int_{-\infty}^{+\infty} f(x, x-z)dx = \int_{-\infty}^{+\infty} f(z+y, y)dy.$$

If X and Y are independent, then

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(x-z)dx = \int_{-\infty}^{+\infty} f_X(z+y) f_Y(y)dy.$$

Continuous Case

(2) Let $Z = X \cdot Y$. Then

$$f_Z(z) = \int_{-\infty}^{+\infty} f(x, z/x)\frac{1}{|x|}dx = \int_{-\infty}^{+\infty} f(z/y, y)\frac{1}{|y|}dy.$$

If X and Y are independent, then

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z/x)\frac{1}{|x|}dx = \int_{-\infty}^{+\infty} f_X(z/y) f_Y(y)\frac{1}{|y|}dy.$$

Continuous Case

(3) Let $Z = X/Y$. Then

$$f_Z(z) = \int_{-\infty}^{+\infty} f(x, x/z)\frac{|x|}{z^2}dx = \int_{-\infty}^{+\infty} f(yz, y)|y|dy.$$

If X and Y are independent, then

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x)f_Y(x/z)\frac{|x|}{z^2}dx = \int_{-\infty}^{+\infty} f_X(yz)f_Y(y)|y|dy.$$

Continuous Case

2. The case of $\max(\boldsymbol{X}, \boldsymbol{Y})$

Set

$$Z = \max(X, Y) = \begin{cases} X & \text{for } X > Y, \\ Y & \text{for } X \leqslant Y. \end{cases}$$

We have

方法：先求F，然后求导

$$\begin{aligned} F_Z(z) &= P(\max(X, Y) \leqslant z) \\ &= P(X \leqslant z, Y \leqslant z) \\ &= F(z, z). \end{aligned}$$

Continuous Case

$$F_Z(z) = P(X \leqslant z, Y \leqslant z) = F(z, z).$$

If X and Y are independent, then

$$F_Z(z) = P(X \leqslant z)P(Y \leqslant z) = F_X(z)F_Y(z) \tag{4}$$

and hence

$$f_Z(z) = F_X(z)f_Y(z) + f_X(z)F_Y(z). \tag{5}$$

Continuous Case

If X and Y are independent and have identical distribution function F and probability density function f, then equation (4) becomes

$$F_Z(z) = [F(z)]^2. \tag{6}$$

Equation (5) becomes

$$f_Z(z) = 2F(z)f(z). \tag{7}$$

3. The case of min($\boldsymbol{X}$, $\boldsymbol{Y}$)

Set

$$W = \min(X, Y) = \begin{cases} Y & \text{for } X > Y, \\ X & \text{for } X \leqslant Y. \end{cases}$$

Thus,

$$\begin{aligned} F_W(w) &= P(\min(X, Y) \leqslant w) \\ &= P(\{Y \leqslant w, X > Y\} \cup \{X \leqslant w, X \leqslant Y\}). \end{aligned}$$

Continuous Case

Since the event $\{\min(X,Y) \leqslant w\}$ contains many cases, we consider its complement. Thus

$$\begin{aligned} F_W(w) &= 1 - P(\min(X,Y) > w) = 1 - P(X > w, Y > w) \\ &= F_X(w) + F_Y(w) - F_{X,Y}(w,w). \end{aligned}$$

If X and Y are independent, then

$$F_W(w) = 1 - P(X > w)P(Y > w) = 1 - [1 - F_X(w)][1 - F_Y(w)]. \tag{8}$$

If X and Y are independent and have the same distribution function F and probability density function f, then

$$F_W(w) = 1 - [1 - F(w)]^2.$$

And

$$f_W(w) = F_W'(w) = 2[1 - F(w)]f(w). \tag{9}$$

Continuous Case

Example

Suppose that $X_1 \sim Exp(\alpha)$, $X_2 \sim Exp(\beta)$ and $X_1 \perp X_2$. Let $Z = \max(X, Y)$ and $W = \min(X, Y)$. Determine the distributions of Z and W.

Solution. Since $X_1 \sim Exp(\alpha)$ and $X_2 \sim Exp(\beta)$,

$$F_X(x) = \begin{cases} 1 - e^{-\alpha x} & \text{for } x > 0, \\ 0 & \text{for } x \leqslant 0 \end{cases} \text{ and } F_Y(y) = \begin{cases} 1 - e^{-\beta y} & \text{for } y > 0, \\ 0 & \text{for } y \leqslant 0. \end{cases}$$

By using equation (4), we get

$$F_Z(z) = F_X(z)F_Y(z) = \begin{cases} (1 - e^{-\alpha z})(1 - e^{-\beta z}) & \text{for } z > 0, \\ 0 & \text{for } z \leqslant 0 \end{cases}$$

Continuous Case

and hence

$$f_Z(z) = \begin{cases} \alpha e^{-\alpha z} + \beta e^{-\beta z} - (\alpha+\beta)e^{-(\alpha+\beta)z} & \text{for } z > 0, \\ 0 & \text{for } z \leqslant 0. \end{cases}$$

By using equation (8), we can obtain

$$F_W(w) = 1 - [1 - F_X(w)][1 - F_Y(w)] = \begin{cases} 1 - e^{-(\alpha+\beta)w} & \text{for } w > 0, \\ 0 & \text{for } w \leqslant 0, \end{cases}$$

and hence

$$f_W(w) = \begin{cases} (\alpha+\beta)e^{-(\alpha+\beta)w} & \text{for } w > 0, \\ 0 & \text{for } w \leqslant 0. \end{cases}$$

i.e., $W \sim Exp(\alpha + \beta)$. □

Thank you for your patience !