Chapter 3 Random Vectors

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Random Vectors

There are two efficient tools to study random vectors.

One is studying the joint distribution, which gives the probability of each possible pair of values for the random vector.

Another is observing numerical characteristics, which can give us information of the relationship between random variables. As you can imagine, to handle the problems of random vectors must be getting more complicated.

Contents

Random Vectors and Joint Distributions

Let X and Y be two random variables defined on the same probability space.

The distribution functions $F_X(x)$ and $F_Y(y)$ of the given random variables only determine their separate (marginal) statistics.

How can we describe their joint statistics?

In this section, we show that the joint statistics of the random variables X and Y can be completely determined by their joint distribution function.

Random Vectors and Joint Distributions

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The ***joint distribution function*** (***joint d.f.***) of X and Y is defined by

$$F(x, y) = P(X \leqslant x, Y \leqslant y), \quad -\infty < x, y < \infty. \tag{1}$$

All joint probability statements about X and Y can, in theory, be answered in terms of their joint distribution function by using the equation (1).

Random Vectors and Joint Distributions

For example,

(i) for any $x_1 < x_2, y_1 < y_2$, the event $\{X \leqslant x, y_1 < Y \leqslant y_2\}$, we maintain that

$$P(X \leqslant x,\ y_1 < Y \leqslant y_2) = F(x, y_2) - F(x, y_1);$$

(ii) for the event $\{x_1 < X \leqslant x_2, Y \leqslant y\}$, we maintain that

$$P(x_1 < X \leqslant x_2, Y \leqslant y) = F(x_2, y) - F(x_1, y);$$

(iii) for the event $\{x_1 < X \leqslant x_2, y_1 < Y \leqslant y_2\}$, we have

$$\begin{aligned} & P(x_1 < X \leqslant x_2, y_1 < Y \leqslant y_2) \\ =& F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1). \end{aligned}$$

Random Vectors and Joint Distributions

Proposition

The joint distribution function $F(x, y)$ *is such that*

(i) *for any* $x_1 < x_2$, $F(x_1, y) \leqslant F(x_2, y)$;

for any $y_1 < y_2$, $F(x, y_1) \leqslant F(x, y_2)$.

(ii) $F(-\infty, y) = 0, \quad F(x, -\infty) = 0, \quad F(\infty, \infty) = 1.$

(iii) $\lim\limits_{x \to x_0^+} F(x, y) = F(x_0, y), \qquad \lim\limits_{y \to y_0^+} F(x, y) = F(x, y_0).$

右连续

A natural question is what values $F(x, +\infty)$ and $F(+\infty, y)$ are.

Random Vectors and Joint Distributions

The d.f. of the single variable X can be derived from the joint d.f. $F(x, y)$ as follows, for $-\infty < x < \infty$,

$$\begin{aligned} F_X(x) &= P(X \leqslant x) = P(X \leqslant x, Y < \infty) = F(x, \infty) \\ &= P(\lim_{y \to \infty} \{X \leqslant x, Y \leqslant y\}) \\ &= \lim_{y \to \infty} P(X \leqslant x, Y \leqslant y) = \lim_{y \to \infty} F(x, y). \end{aligned}$$

$$i.e., \quad F_X(x) = \lim_{y \to \infty} F(x, y).$$

Similarly, if $F_Y(y)$ denotes the $d.f.$ of Y, then for $-\infty < y < \infty$,

$$F_Y(y) = \lim_{x \to \infty} F(x, y).$$

The $d.f.$ F_X and F_Y are referred to as the ***marginal distributions*** of X and Y.

Random Vectors and Joint Distributions

For instance, we wanted to compute the joint probability that X is greater than x and Y is greater than y. This could be done as follows.

$$
\begin{aligned}
&P(X > x, Y > y) \\
&= 1 - P(\overline{\{X > x, Y > y\}}) \\
&= 1 - p(\overline{\{X > x\}} \cup \overline{\{Y > y\}}) \\
&= 1 - P(\{X \leqslant x\} \cup \{Y \leqslant y\}) \\
&= 1 - [P(X \leqslant x) + P(Y \leqslant y) - P(X \leqslant x, Y \leqslant y)] \\
&= 1 - F_X(x) - F_Y(y) + F(x, y).
\end{aligned}
$$

Random Vectors and Joint Distributions

Example

The joint distribution function of X and Y is

$$F(x,y) = A\left(B + \arctan\frac{x}{2}\right)\left(C + \arctan\frac{y}{2}\right), \quad -\infty < x, y < \infty.$$

(a) Determine the values of the constants A, B and C.
(b) Determine the value of $P(0 < X \leqslant 2,\ 0 < Y \leqslant 3)$.
(c) Find the marginal distributions of X and Y.

Solution. (a) Since $F(\infty, \infty) = 1$,

$$F(\infty, \infty) = A(B + \pi/2)(C + \pi/2) = 1.$$

Since $F(-\infty, y) = 0$ and $F(x, -\infty) = 0$,

$$F(-\infty, y) = A(B - \pi/2)\left(C + \arctan \frac{y}{2}\right) = 0$$

and

$$F(x, -\infty) = A\left(B + \arctan \frac{x}{2}\right)(C - \pi/2) = 0.$$

Thus we have $B = C = \dfrac{\pi}{2}$ and $A = \dfrac{1}{\pi^2}$.

(b) From equation (3.2), we know

$$P(0 < X \leqslant 2,\ 0 < Y \leqslant 3)$$

$$= F(0, 0) + F(2, 3) - F(0, 3) - F(2, 0) = 1/16.$$

Solution. (c) The marginal distributions of X and Y are

$$F_X(x) = \lim_{y\to\infty} F(x,y) = \frac{1}{\pi}\Big(\frac{\pi}{2} + \arctan\frac{x}{2}\Big),$$

and

$$F_Y(y) = \lim_{x\to\infty} F(x,y) = \frac{1}{\pi}\Big(\frac{\pi}{2} + \arctan\frac{y}{2}\Big)$$

respectively. $\square$

Contents

Discrete Random Vectors

For discrete bivariate random vectors, we always use the ***joint probability function*** (joint $p.f.$)

$$p(x, y) := P(\{X = x\} \cap \{Y = y\}) \equiv P(X = x, Y = y)$$

of the pair of discrete random vector (X, Y), whose possible values are a (finite or countably infinite) set of points (x_j, y_k) in the plane, where $p(x, y)$ has the following properties:

(i) $p(x_j, y_k) \geqslant 0$, $\forall (x_j,\ y_k) \in \mathbb{R} \times \mathbb{R}$;

(ii) $\displaystyle\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} p(x_j, y_k) = 1.$

Discrete Random Vectors

From the definition of $p(x, y)$, we know

$$F(x, y) = \sum_{x_j \leqslant x} \sum_{y_k \leqslant y} p(x_j, y_k). \tag{2}$$

The probability $P[(X, Y) \in A]$ is obtained by summing the joint p.f. over pairs in A:

$$P[(X, Y) \in A] = \sum_{(x_j, y_k) \in A} p(x_j, y_k).$$

When the function $p(x, y)$ is summed over all possible values of Y (resp., X), the resulting function is called the ***marginal probability function*** of X (resp., Y). That is,

$$p_X(x) = \sum_y p(x, y) \quad \text{and} \quad p_Y(y) = \sum_x p(x, y). \tag{3}$$

Discrete Random Vectors

We can describe ***joint probability table*** by using a tabular. The marginal probabilities are described in the last row and the last column.

$y \backslash x$	x_1	x_2	$\cdots$	x_i	$\cdots$	$p_Y(y)$
y_1	$p(x_1, y_1)$	$p(x_2, y_1)$	$\cdots$	$p(x_i, y_1)$	$\cdots$	$p_Y(y_1)$
y_2	$p(x_1, y_2)$	$p(x_2, y_2)$	$\cdots$	$p(x_i, y_2)$	$\cdots$	$p_Y(y_2)$
$\vdots$	$\ldots$	$\ldots$	$\ddots$	$\ldots$	$\ldots$	$\vdots$
y_j	$p(x_1, y_2)$	$p(x_2, y_2)$	$\cdots$	$p(x_i, y_j)$	$\cdots$	$p_Y(y_j)$
$\vdots$	$\ldots$	$\ldots$	$\ddots$	$\ldots$	$\ldots$	$\vdots$
$p_X(x)$	$p_X(x_1)$	$p_X(x_2)$	$\cdots$	$p_X(x_i)$	$\cdots$	1

Table: The joint probability table of X and Y

Example

Suppose that the joint probability table of X and Y is described in

$y \backslash x$	-1	0	2
0	0.1	0.3	a
1	0.1	0.1	b

Assume $P(XY = 0) = 0.7$. Find the values of a and b.

Solution. Since $P(XY = 0) = 0.7$,

$$\begin{aligned} P(XY = 0) &= P(\{X = 0\} \cup \{Y = 0\}) \\ &= P(X = 0) + P(Y = 0) - P(X = 0, Y = 0) \\ &= 0.4 + 0.4 + a - 0.3 = 0.7 \end{aligned}$$

Discrete Random Vectors

Thus $a = 0.2$. By the property $\sum_{(x,y)} p(x,y) = 1$, i.e., $a + b = 0.4$, we get $b = 0.2$. $\square$

Example

The distributions of X and Y are given in following Table.

x	-1	0	1
$p_X(x)$	$1/4$	$1/2$	$1/4$

y	0	1
$p_Y(y)$	$1/2$	$1/2$

Suppose that $P(XY = 0) = 1$. Find the joint probability function of X and Y.

Solution. Since $P(XY=0)=1$, $p_X(0)=1/2$, and $p_Y(0)=1/2$,

$$P(XY=0)=P(\{X=0\}\cup\{Y=0\})$$

$$=p_X(0)+p_Y(0)-p(0,0)=1-p(0,0)=1.$$

Thus $p(0,0)=0$, i.e., the joint probability table of X and Y is given in Table

$y \backslash x$	-1	0	1	$p_Y(y)$
0	$p(-1,0)$	0	$p(1,0)$	$1/2$
1	$p(-1,1)$	$p(0,1)$	$p(1,1)$	$1/2$
$p_X(x)$	$1/4$	$1/2$	$1/4$	1

.

Discrete Random Vectors

By using equation (3), we immediately obtain

$$\begin{aligned}
&0 + p(0,1) = 1/2, \\
&p(-1,1) + p(0,1) + p(1,1) = 1/2, \\
&p(-1,0) + p(-1,1) = 1/4, \\
&p(1,0) + p(1,1) = 1/4.
\end{aligned}$$

Since $p(x,y) \geqslant 0$, we obtain

$$p(0,1) = 1/2, \quad p(-1,1) = p(1,1) = 0, \quad p(-1,0) = p(1,0) = 1/4.$$

Thus

$y \backslash x$	-1	0	1	$p_Y(y)$
0	$1/4$	0	$1/4$	$1/2$
1	0	$1/2$	0	$1/2$
$p_X(x)$	$1/4$	$1/2$	$1/4$	1

$\square$

Discrete Random Vectors

We have known that the d.f. of a discrete random variable X is a step function, and all possible values are exactly those breakpoints of the step function.

How about the relationship between d.f. and p.f. of discrete bivariate random vectors (X, Y)?

Theoretically, we have equation (2), but it is really not that easy.

Example

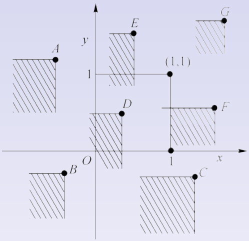
Suppose that the joint probability table of X and Y is Table 2. Find the joint d.f. of X and Y.

$y \backslash x$	0	1
0	1/2	0
1	0	1/2

Table: The joint probability table of X and Y in Example 4

Discrete Random Vectors

Solution. From the following Fig.,



we observe that for those kind of points A, B and C, we have $F(x, y) = P(X \leqslant x, Y \leqslant y) = P(\emptyset) = 0$. For the kind of points D, E and F in first quadrant,

$$F(x, y) = P(X \leqslant x, Y \leqslant y) = P(X = 0, Y = 0) = 1/2.$$

Discrete Random Vectors

For the point G, which is in the upper right corner of the point (1,1),

$$F(x, y) = P(X \leqslant x, Y \leqslant y)$$

$$= P(\{X = 0, Y = 0\} \cup \{X = 1, Y = 1\}) = 1/2 + 1/2 = 1.$$

So we have

$$F(x, y) = \begin{cases} 0 & \text{for } x < 0 \text{ or } y < 0, \\ 1/2 & \text{for } 0 \leqslant x < 1,\ y \geqslant 0 \text{ or } x \geqslant 1,\ 0 \leqslant y < 1, \\ 1 & \text{for } x \geqslant 1, y \geqslant 1. \end{cases}$$

Contents

Continuous Random Vectors

Let X and Y be two continuous random variables. The random variables X and Y are called ***(jointly) continuous*** if the joint distribution function $F(x, y)$ can be expressed by the ***joint probability density function*** (joint p.d.f.) $f(x, y)$ as

$$F(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(u, v) du dv.$$

The joint p.d.f. $f(x, y)$ has the following properties:

(i) $f(x, y) \geqslant 0$ for any point $(x, y) \in \mathbb{R} \times \mathbb{R}$;

(ii) $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) dx dy = 1.$

Continuous Random Vectors

The probability $P[(X,Y) \in A]$ is obtained by integrating the joint p.d.f. over the domain A:

$$P[(X,Y) \in A] = \iint_{(x,y)\in A} f(x,y)dxdy.$$

Let $F(x,y)$ be a joint distribution. Suppose that $\dfrac{\partial^2 F}{\partial x \partial y}$ exists and is nonnegative, except possibly on a finite collection of lines in $\mathbb{R}^2$. Then the function $f(x,y)$ can be obtained by

$$f(x,y) = \begin{cases} \dfrac{\partial^2 F}{\partial x \partial y} & \text{where this exists,} \\ 0 & \text{elsewhere.} \end{cases}$$

Example

A bank operates both a drive-up facility and a walk-up window. On a randomly selected day, let X be the proportion of time that the drive-up facility is in use (at least one customer is being served or waiting to be served) and Y be the proportion of time that the walk-up window is in use. Then the set of possible values for (X, Y) is the rectangle $D = \{(x, y) : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$. Suppose that the joint p.d.f. of (X, Y) is given by

$$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & \text{for } 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Verify that $f(x, y)$ is a legitimate joint p.d.f..
(b) Calculate the value of $P(0 \leqslant X \leqslant \frac{1}{4}, 0 \leqslant Y \leqslant \frac{1}{4})$.

Solution. (a) To verify that this is a legitimate p.d.f., note that $f(x, y) \geqslant 0$ and

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) dx dy \\
&= \int_0^1 \int_0^1 \frac{6}{5}(x + y^2) dx dy \\
&= \int_0^1 \int_0^1 \frac{6}{5} x dx dy + \int_0^1 \int_0^1 \frac{6}{5} y^2 dx dy \\
&= \int_0^1 \frac{6}{5} x dx + \int_0^1 \frac{6}{5} y^2 dy = \frac{6}{10} + \frac{6}{15} = 1.
\end{aligned}$$

(b) The desired probability is

$$
\begin{aligned}
&P(0 \leqslant X \leqslant \frac{1}{4}, 0 \leqslant Y \leqslant \frac{1}{4}) \\
=&\int_0^{1/4}\int_0^{1/4}\frac{6}{5}(x+y^2)dxdy \\
=&\int_0^{1/4}\int_0^{1/4}\frac{6}{5}xdxdy+\int_0^{1/4}\int_0^{1/4}\frac{6}{5}y^2dxdy \\
=&\frac{7}{640}=0.0109.
\end{aligned}
$$

$\square$

Continuous Random Vectors

Example

Let X and Y have density

$$f(x,y) = \begin{cases} 8xy & \text{if } 0 < y < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

What are $P(2X > 1, 2Y < 1)$ and $P(X + Y > 1)$? Find $F(x,y)$.

Solution. Notice that constraints $2X > 1$, $2Y < 1$ require that $(X,Y) \in A$, where A is the square with vertices $(1/2, 0), (1, 0), (1/2, 1/2), (1, 1/2)$ of Fig. 3.5(a). Hence,

$$P(2X > 1, 2Y < 1) = \int_{\frac{1}{2}}^{1} \int_{0}^{\frac{1}{2}} f(x,y) dy dx$$

$$= 8 \int_{\frac{1}{2}}^{1} x dx \int_{0}^{\frac{1}{2}} y dy = \frac{3}{8}.$$

Continuous Random Vectors

Likewise, $X+Y>1$ if $(X,Y)\in B$, where B is the triangle with vertices $(1/2,0),(1,0),(1,1)$ of Fig. 3.5(b). Hence,

$$P(X+Y>1)=\iint\limits_{x+y>1,\ y<x} 8xydxdy=8\int_{\frac{1}{2}}^{1}dx\int_{1-x}^{x}xydy=\frac{5}{6}.$$

Finally, for the points A,B and C satisfying $x\leqslant 0$ or $y\leqslant 0$, $F(x,y)=0$.
For the point D satisfying $0<y<x<1$,

$$F(x,y)=\int_0^y dv\int_v^x 8uvdu=2x^2y^2-y^4.$$

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For the point E satisfying $0 < x < 1, y \geqslant x$,

$$F(x,y) = \int_0^x du \int_0^u 8uvdv = x^4.$$

For the point F satisfying $x \geqslant 1, 0 \leqslant y < 1$,

$$F(x,y) = \int_0^y dv \int_v^1 8uvdu = 2y^2 - y^4.$$

For the point (x,y) satisfying $x > 1, y \geqslant 1$, $F(x,y) = 1$. As a result,

$$F(x,y) = \begin{cases} 0 & \text{for } x \leqslant 0 \text{ or } y \leqslant 0, \\ 2x^2y^2 - y^4 & \text{for } 0 < x < y < 1, \\ x^4 & \text{for } 0 < x < 1, y \geqslant x, \\ 2y^2 - y^4 & \text{for } x \geqslant 1, 0 \leqslant y < 1, \\ 1 & \text{for } x > 1, y \geqslant 1. \end{cases}$$

□

Continuous Random Vectors

For continuous random vector (X, Y), we have

$$F_X(x) = \int_{-\infty}^{x} \int_{-\infty}^{+\infty} f(u, v) dv du,$$

$$F_Y(y) = \int_{-\infty}^{y} \int_{-\infty}^{+\infty} f(u, v) du dv$$

since $F_X(x) = \lim_{y \to \infty} F(x, y)$ and $F_Y(y) = \lim_{x \to \infty} F(x, y)$.

By differentiating, we get

Definition

The ***marginal probability density functions*** *of X and Y are defined by*

$$f_X(x) = \int_{-\infty}^{+\infty} f(x, y) dy \quad \text{and} \quad f_Y(y) = \int_{-\infty}^{+\infty} f(x, y) dx. \tag{4}$$

Example

Consider the joint p.d.f. $f(x, y)$ defined by

$$f(x, y) = cxye^{-x^2-y^2} \quad \text{for } x \geqslant 0, y \geqslant 0,$$

where $c > 0$ is a constant.
(a) Determine the constant c.
(b) Determine the joint d.f. of (X, Y).
(c) Determine the marginal density function of X.

Solution. (a) Since $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy = 1$,

$$\int_0^{+\infty}\int_0^{+\infty} cxye^{-x^2-y^2}dxdy = c\int_0^{+\infty} xe^{-x^2}dx\int_0^{+\infty} ye^{-y^2}dy$$
$$= c\Big[-\frac{e^{-x^2}}{2}\Big|_0^{\infty}\Big]\Big[-\frac{e^{-y^2}}{2}\Big|_0^{\infty}\Big] = c(1/2)(1/2) = c/4.$$

Thus this function is a valid joint density fuction if and only if the constant c is equal to 4.

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(b) The joint distribution function of the pair (X, Y) is given by

$$\begin{aligned} F(x,y) &= \int_0^y \int_0^x 4uve^{-u^2-v^2} du dv \\ &= \int_0^x 2ue^{-u^2} du \int_0^y 2ve^{-v^2} dv \\ &= \Big(-e^{-u^2} \Big|_0^x \Big) \Big(-e^{-v^2} \Big|_0^y \Big) = (1-e^{-x^2})(1-e^{-y^2}) \end{aligned}$$

for $x \geqslant 0, y \geqslant 0$.

(c) The marginal density function of X is

$$\begin{aligned} f_X(x) &= \int_0^{+\infty} 4xye^{-x^2-y^2} dy = 2xe^{-x^2} \int_0^{+\infty} 2ye^{-y^2} dy \\ &= 2xe^{-x^2} \Big(-e^{-y^2} \Big|_0^{+\infty} \Big) = 2xe^{-x^2} \qquad \text{for } x \geqslant 0. \end{aligned}$$

□

Continuous Random Vectors

Example

Suppose that (X, Y) are random vector that can only take values in the interval $[0, 2]$. Suppose that also that the joint d.f. of X and Y, for $0 \leqslant x \leqslant 2$ and $0 \leqslant y \leqslant 2$, as follows:

$$F(x, y) = \frac{1}{16} xy(x + y).$$

Determine the marginal p.d.f. $f_X(x)$ of the random variable X.

Solution. Since (X, Y) takes value only in the interval $[0, 2]$, we know if either $x < 0$ or $y < 0$, then $F(x, y) = 0$. If both $x > 2$ and $y > 2$, then $F(x, y) = 1$. If $0 \leqslant x \leqslant 2$ and $y > 2$, then $F(x, y) = F(x, 2)$, and it means

$$F(x, y) = \frac{1}{8} x(x + 2).$$

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Similarly, if $0 \leqslant y \leqslant 2$ and $x > 2$, then

$$F(x, y) = \frac{1}{8} y(y+2).$$

The function $F(x, y)$ has now been specified for every point in the xy-plane.
By letting $y \to \infty$, we find that the d.f. of the random variable X is

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0, \\ \frac{1}{8} x(x+2) & \text{for } 0 \leqslant x \leqslant 2, \\ 1 & \text{for } x > 2. \end{cases}$$

By differentiating,

$$f_X(x) = \begin{cases} \dfrac{x+1}{4} & \text{for } 0 \leqslant x \leqslant 2, \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

Example

(Uniform distribution) Suppose that you pick a point (X, Y) at random in the rectangle $D = \{(x, y) : 0 < x < a, 0 < y < b\}$. That is, if A is a subset of D with area $|A|$, then we have

$$P((X, Y) \in A) = \frac{|A|}{ab}.$$

We call (X, Y) has uniform distribution on the domain D, denoted by $(X, Y) \sim U(D)$.
(a) Find the joint d.f. and p.d.f. of (X, Y).
(b) Find the marginal p.d.f. of X and Y respectively.

Solution. (a) If $x \geqslant a, y \geqslant b$, then

$$F(x, y) = P(0 \leqslant X \leqslant a, 0 \leqslant Y \leqslant b) = \frac{ab}{ab} = 1.$$

If $0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b$, then

$$F(x, y) = P(0 \leqslant X \leqslant x, 0 \leqslant Y \leqslant y) = \frac{xy}{ab}.$$

If $x \geqslant a,\ 0 \leqslant y \leqslant b$, then

$$F(x, y) = P(0 \leqslant X \leqslant a, 0 \leqslant Y \leqslant y) = \frac{ay}{ab} = \frac{y}{b}.$$

Similarly, if $0 \leqslant x \leqslant a,\ y \geqslant b$, then $F(x, y) = x/a$. Otherwise, $F(x, y) = 0$. That is,

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$$F(x,y) = \begin{cases} 1 & \text{for } x \geqslant a, y \geqslant b, \\ \dfrac{xy}{ab} & \text{for } 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b, \\ \dfrac{y}{b} & \text{for } x \geqslant a,\ 0 \leqslant y \leqslant b, \\ \dfrac{x}{a} & \text{for } 0 \leqslant x \leqslant a,\ y \geqslant b, \\ 0 & \text{elsewhere.} \end{cases}$$

Differentiating wherever possible gives the joint p.d.f. of (X, Y)

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y} = \begin{cases} \dfrac{1}{ab} & \text{for } 0 < x < a,\ 0 < y < b, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

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(b) By equation (4), if $0 < x < a$, then

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)dy = \int_0^b \frac{1}{ab} dy = \frac{1}{a}.$$

Otherwise, $f_X(x) = 0$. i.e., $X \sim U(0,a)$,

$$f_X(x) = \begin{cases} \dfrac{1}{a} & \text{for } 0 < x < a, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, $Y \sim U(0,b)$,

$$f_Y(y) = \begin{cases} \dfrac{1}{b} & \text{for } 0 < y < b, \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

Continuous Random Vectors

In fact, the domain D in above example can be generalized to any finite region on $\mathbb{R}^2$. If $(X, Y) \sim U(D)$, then for any subregion $A \in D$,

$$P((X,Y) \in A) = \frac{|A|}{|D|},$$

where $|D|$ is the area of D. Obviously, the probability is only dependent on the subregion's area and independent of its position or its shape. The corresponding joint p.d.f. is

$$f(x,y) = \begin{cases} \dfrac{1}{|D|} & \text{if } (x,y) \in D, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Example

If the joint p.d.f. of (X, Y) is specified by

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]}, \tag{7}$$

$$-\infty < x < +\infty, \quad -\infty < y < +\infty,$$

where $-\infty < \mu_1 < +\infty$, $-\infty < \mu_2 < +\infty$, $\sigma_1 > 0$, $\sigma_2 > 0$, $-1 < \rho < 1$, then we say that (X, Y) has a ***bivariate normal distribution***, and denote it by $(X, Y) \sim N(\mu_1, \mu_2; \sigma_1^2, \sigma_2^2; \rho)$. Find the marginal p.d.f. of X and Y.

Solution. Let $t = \rho\frac{x-\mu_1}{\sigma_1\sqrt{1-\rho^2}} - \frac{y-\mu_2}{\sigma_2\sqrt{1-\rho^2}}$. Then

$$-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(x-\mu_2)^2}{\sigma_2^2}\right] = -\frac{t^2}{2} - \frac{(x-\mu_1)^2}{2\sigma_1^2}.$$

Continuous Random Vectors

By equation (4),

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{+\infty} f(x,y)dy \\
&= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_{-\infty}^{+\infty} \exp\Big\{ -\frac{1}{2}\Big[\rho\frac{x-\mu_1}{\sigma_1\sqrt{1-\rho^2}} - \frac{y-\mu_2}{\sigma_2\sqrt{1-\rho^2}}\Big]^2 - \frac{(x-\mu_1)^2}{2\sigma_1^2}\Big\}dy \\
&= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_{-\infty}^{+\infty} \exp\Big\{ -\frac{t^2}{2} - \frac{(x-\mu_1)^2}{2\sigma_1^2}\Big\} \cdot \big(-\sigma_2\sqrt{1-\rho^2}\big)dt \\
&= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}}\, dt \\
&= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1}}.
\end{aligned}
$$

It is exactly a normal distribution with parameters μ_1 and σ_1^2, i.e., $X \sim N(\mu_1, \sigma_1^2)$. Similarly, we can prove $Y \sim N(\mu_2, \sigma_2^2)$. $\square$

Thank you for your patience !