Section 1.3 Probabilities Defined on Events

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Classical Probability

Definition of Classical Experiment

The classical definition or interpretation of probability is identified with the works of Jacob Bernoulli and Pierre-Simon Laplace. It is based on the concept of equally likely outcomes.

Generally, a random experiment E is ***classical*** if

1. E contains only different limited basic events, that is,

$$\Omega = \{\omega_1, \omega_2, \cdots, \omega_n\}.$$

   We call this kind of sample space ***simple space***, and
2. all outcomes are equally likely to occur.

For example, when a fair coin is tossed, there are two possible outcomes: a head or a tail. A head and a tail are equally likely to occur.

Classical Probability

Definition of Classical Probability

For classical random experiment E, the corresponding problems of probability belong to the category of ***classical probability***.

Definition

For classical random experiment E, $\Omega = \{\omega_1, \omega_2, \cdots, \omega_n\}$, we define the probability of event A as

$$P(A) = \frac{\#A}{\#\Omega} \tag{1}$$

where $\#A$ means the number of all possible outcomes of event A, $\#\Omega$ means the number of all possible outcomes of sample space Ω. If A is comprised by k different elementary events, then

$$P(A) = \frac{k}{n}.$$

Geometric Probability

Example

what is the probability that a point is selected randomly from the interval $[0, 1]$? If your answer is 0, how can you get that? Is it coming from $\frac{1}{\infty}$? If so, what is the probability of choosing the interval $[0, 0.5)$? Is your answer $\frac{1}{2}$? Obviously, we can not get that by classical probability.

The definition of classical probability can not be used any more when the concept of equally likely outcomes is about to be extended to a line or 2D or 3D area.

Geometric Probability

A random experiment E is called to be ***geometric*** if

(i) the sample space is a measurable (such as length, area, volume, etc.) region, i.e., $0 < L(\Omega) < \infty$, and
(ii) the probability of every event $A \subset \Omega$ is proportional to the measure $L(A)$ and has nothing to do with its position and shape.

In this case, we define the probability of event A as

$$P(A) = \frac{L(A)}{L(\Omega)}$$

and $P(\emptyset) = 0$.

The Frequency Interpretation of Probability

Suppose that some random procedure has several possible outcomes that are not necessarily equally likely.

How can we define the probability $P(A)$ of any eventuality A of interest?

For example, suppose that the procedure is the rolling of a die that is suspected to be weighted, or even clearly asymmetrical, in not being a perfect cube. What now is the probability of a six?

The Frequency Interpretation of Probability

Let E be an random experiment, A be an random event. Suppose that E was repeated n times under similar conditions. Let $f_n(A)$ be the times that A occurs. The ration

$$F_n(A) = \frac{f_n(A)}{n}$$

is said to be the ***frequency*** of event A in the n trials. If n is large enough, the probability of event A will be approximated by $F_n(A)$.

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A Lot of Examples

Example

Suppose that an urn contains α white balls and β black balls, and that $k+1$ $(k+1 \leqslant \alpha+\beta)$ balls are to be selected consecutively at random without replacement. Try to determine the probability that the last selected ball is exactly white ball.

Example

Suppose that there are n people, each person will be assigned to any of the $N(n \leqslant N)$ rooms with the same probability $1/N$. We shall determine the probabilities of the following events:

A : For the given n rooms, there is exactly one person in one room.

B : There is exactly one person in one room.

C : There are exactly m people in a given room.

A Lot of Examples

Example

(**Birthday problem**) If a group consists of n people, what is the probability that at least two of them have the same birthday? Ignore leap years and assume that each day in the year is equally likely as a birthday.

Example

A reception has received 12 visits in a week. Suppose that all 12 receptions are proceeded on Tuesday and Thursday. Is the reception time required?

Solution. Assume that the reception time is not specified. Then, the probability of the event that all 12 receptions are proceeded on Tuesday and Thursday is

$$\frac{2^{12}}{7^{12}} = 0.0000003.$$

This is a very small probability. Practical experience shows that rare event should seldom occur (referred to as the impossible principle). Now in this example, the rare events have happened in one experiment. So there is a reason to doubt the validity of the assumption and we can reach the conclusion that the reception time is required. □

A Lot of Examples

Example

(**Lunch date problem**) You and one of your friends arrange to meet between 12:00 and 13:00. As a result, it is possible for one of you to arrive at random between 12:00 and 13:00 and waits exactly 20 minutes for another one. After 20 minutes, one of you leaves if another person has not arrived. What is the probability that you and your friend will meet?

Solution. Let x and y be the time you and your friend arriving the gate, respectively, then our sample space is a square $\Omega = \{(x, y) |\ 12 \leqslant x, y \leqslant 13\}$. Let M be the event that two of you will meet at the gate, then M will occur if and only if $|x - y| \leqslant 1/3$, i.e.,

$$M = \{(x, y) |\ 12 \leqslant x, y \leqslant 13,\ |x - y| \leqslant 1/3\}.$$

A Lot of Examples

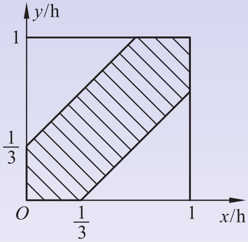

Figure: Lunch date problem

In this problem, the expression “equally likely to occur” means that the probability that the sample point is located in a special region $M \subset \Omega$ is proportional to the area of M. Again, since the certain event Ω has area 1, the probability of M is equal to the area of M, i.e., $P(M) = 1 - \left(\frac{2}{3}\right)^2 = \frac{5}{9}$. $\square$

A Lot of Examples

Example

(**Buffon's needle problem**) Given a needle of length l dropped on a plane ruled with parallel lines d $(l < d)$ units apart. What is the probability that the needle will cross a line?

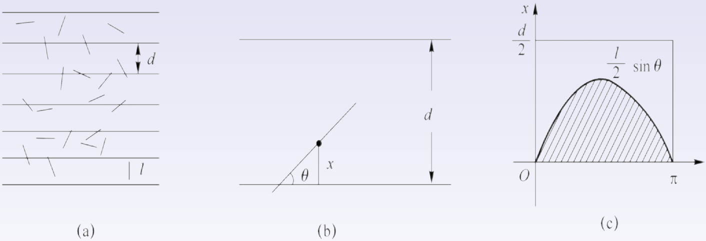

Figure: Buffon's needle problem

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Problems of Classical Probability

Obvious!

Problems of Geometric Probability

Is the definition of geometric probability perfect?

Let us see the following famous example.

Example

(**Bertrand paradox**) Consider an equilateral triangle inscribed in a circle. Suppose a chord of the circle is chosen at random. What is the probability that the chord is longer than a side of the triangle? Bertrand gave three arguments, all apparently valid, yet yielding different results.

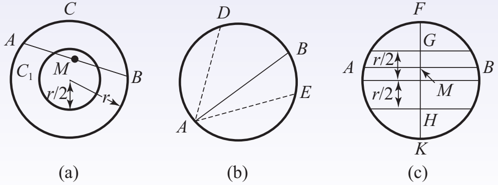

Figure: Bertrand paradox

Problems of Frequency

Is this interpretation perfect?

(i) there is no definite indication of an actual number that would be considered large enough;

(ii) this interpretation of probability rests on the important assumption that our process or experiment can be repeated many times under similar conditions. While the real-world experiment must not be completely controlled but must have some "random" features;

(iii) the probability of event A will be approximated by $F_n(A)$ do not mean $P(A)$ is the limit of $F_n(A)$;

Problems of The Frequency

(iv) the frequency interpretation of probability is that it applies only to a problem in which there can be, at least in principle, a large number of similar repetition of a certain process. Many important problems are not of this type.

For example, the frequency interpretation of probability cannot be applied directly to the probability that a specific acquaintance will get married within the next two years or to the probability that a particular medical research project will lead to the development of a new treatment for a certain disease within a specified period of time.

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Similarity of the Above Interpretations of probability

Theorem

For classical random experiment E, the probability has the following properties:
(i) *for every event A,* $P(A) \geqslant 0$,
(ii) $P(\Omega) = 1$,
(iii) *for every finite sequence of n disjoint events* $A_1, A_2, \cdots, A_n$,

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i).$$

Property (iii) *is called finite additivity.*

Geometric Probability

Theorem

For Geometrical random experiment E, the probability has the following properties:
(i) *for every event A, $P(A) \geqslant 0$,*
(ii) $P(\Omega) = 1$, *and*
(iii) ***for every countable disjoint events*** $A_1, A_2, \cdots$,

$$P\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} P(A_i).$$

Property (iii) is called countable additivity.

Geometric Probability

Proof. (i) and (ii) are obvious.
(iii) According to countable additivity in measure theory, i.e., for every countable disjoint sequence $A_1, A_2, \cdots$,

$$L(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} L(A_i),$$

then

$$P(\bigcup_{i=1}^{\infty} A_i) = \frac{L(\bigcup\limits_{i=1}^{\infty} A_i)}{L(\Omega)} = \sum_{i=1}^{\infty} \frac{L(A_i)}{L(\Omega)} = \sum_{i=1}^{\infty} P(A_i).$$

$\square$

Summary

- **classical probability**:
  - E contains only different limited basic events,
  - all outcomes are equally likely to occur.
- **geometrical probability**:
  - the sample space is a measurable region,
  - the probability of every event $A \subset \Omega$ is proportional to the measure $L(A)$ and has nothing to do with its position and shape.
- **frequency**: do experiments

Each interpretation of probability, as we can see, has its appeal and its difficulties. We need a pure mathematical definition for probability for general cases.

Thank you for your patience !