Section 1.6 Independent of Events

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Independence of Two Events

Sometimes an event can affect the next event. We call those Dependent Events, because what happens depends on what happened before.

Consider the following examples:

(i) Landing on heads after tossing a coin AND rolling a 5 on a single 6-sided die.

(ii) Choosing a marble from a jar AND landing on heads after tossing a coin.

(iii) Choosing a 3 from a deck of cards, replacing it, AND then choosing an ace as the second card.

(iv) Rolling a 4 on a single 6-sided die, AND then rolling a 1 on a second roll of the die.

Independent Events are not affected by previous events.

This is an important idea!

Independence of Two Events

The "operational independence" described indicates that knowledge that one of the events has occurred does not affect the likelihood that the other will occur. For a pair of events A and B, this is the condition

$$P(A|B) = P(A) \text{ or } P(B|A) = P(B).$$

Occurrence of the event A is not "conditioned by" occurrence of the event B. If the occurrence of B does not affect the likelihood of the occurrence of A, we should be inclined to think of the events A and B as being independent in a probability sense.

Independence of Two Events

We take our clue from the condition $P(A|B) = P(A)$ to define the independence of two events.

Definition

Two events A and B are ***independent*** *if*

disjoint: $AB = \emptyset$

$$P(AB) = P(A)P(B). \tag{1}$$

It is obvious that if $P(A) > 0$ and $P(B) > 0$, then independence is equivalent to the statement that the conditional probability of one event given the other is the same as the unconditional probability of the event:

$$P(A|B) = P(A) \Leftrightarrow P(B|A) = P(B) \Leftrightarrow P(AB) = P(A)P(B).$$

Independence of Two Events

The independence of the events in the examples at the beginning of this section are obvious. But things do not always like those.

Example

Let three fair coins be tossed.
Let $A = \{$ all heads or all tails $\}$, $B = \{$ at least two heads $\}$, and $C = \{$ at most two tails $\}$. Of the pairs of events, (A, B), (A, C), and (B, C), which are independent and which are dependent?

Solution. The events can be written explicitly:

$$\begin{aligned} A =& \{HHH, TTT\}, \\ B =& \{HHH, HHT, HTH, THH\}, \\ C =& \{HHH, HHT, HTH, THH, HTT, THT, TTH\}. \end{aligned}$$

Solution. Then

$$
\begin{aligned}
P(AB) &= 1/8 = (2/8)(4/8) = P(A) \cdot P(B), \\
P(AC) &= 1/8 \neq (2/8)(7/8) = P(A) \cdot P(C), \\
P(BC) &= 4/8 \neq (4/8)(7/8) = P(B) \cdot P(C).
\end{aligned}
$$

Thus A and B are independent. A and C are dependent. B and C are dependent. $\square$

Independence of Two Events

The terms independent and disjoint sound vaguely similar but they are actually very different.

Theorem

Suppose that A and B are disjoint events for an experiment, each with positive probability. Then A and B are dependent.

Proof. Since A and B are disjoint events, $P(AB) = P(\emptyset) = 0$. While $P(A)P(B) > 0$ because $P(A) > 0, P(B) > 0$. So $P(AB) \neq P(A)P(B)$. $\square$

Independence of Two Events

Theorem

If A and B are independent events in an experiment, then each of the following pairs of events is independent: (i) $\overline{A}$ *and* B; (ii) A *and* $\overline{B}$; (iii) $\overline{A}$ *and* $\overline{B}$.

Proof. We prove case (i). Suppose that A and B are independent. Then by the difference rule,

$$\begin{aligned} P(\overline{A}B) =& P(B) - P(AB) = P(B) - P(A)P(B) \\ =& [1 - P(A)]P(B) = P(\overline{A})P(B). \end{aligned}$$

Hence $\overline{A}$ and B are independent.
(ii) and (iii) can be obtained by the similar way. □

Independence of Two Events

Example

Let A and B be independent events with $P(A) = 1/4$ and $P(A \cup B) = 2P(B) - P(A)$. Find the values of $P(B)$, $P(A|B)$ and $P(\overline{B}|A)$.

Solution. Since A and B are independent,

$$P(A \cup B) = P(A) + P(B) - P(AB) = P(A) + P(B) - P(A)P(B).$$

Thus,

$$1/4 + P(B) - (1/4)P(B) = 2P(B) - 1/4.$$

This implies that $P(B) = 2/5$.
Since A and B are independent,

$$P(A|B) = P(A) = 1/4 \quad \text{and} \quad P(\overline{B}|A) = P(\overline{B}) = \frac{3}{5}.$$

$\square$

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Independence of Several Events

How to define the independence of three events A, B, C? How to express the occurrence of any event of the three events has nothing to do with the occurrence of the other two events?

Definition

Three events A, B and C in the sample space S of a random experiment are said to be ***mutually independent*** *(or independent) if*

$$
\begin{aligned}
P(AB) &= P(A)P(B), \\
P(AC) &= P(A)P(C), \\
P(BC) &= P(B)P(C), \\
P(ABC) &= P(A)P(B)P(C).
\end{aligned}
$$

Events A, B and C are said to be ***pairwise independent*** *if the first three equations hold.*

Independence of Several Events

Example

Suppose that a balanced coin is independently tossed two times. Define the following events:
A: Head appears on the first toss;
B: Head appears on the second toss;
C: Both tosses yield the same outcome.
Are events A, B and C mutually independent?

Solution. The sample space Ω consists of the following outcomes: $\Omega = \{HH, HT, TH, TT\}$ and the events listed above are $A = \{HH, HT\}$, $B = \{TH, HH\}$, $C = \{HH, TT\}$.
Since the coin is balanced, that is, all outcomes are assigned the same probability, namely $1/4$, we get

Independence of Several Events

Solution.

$$
\begin{aligned}
&P(A) = P(B) = P(C) = 2/4 = 1/2, \\
&P(AB) = P(AC) = P(BC) = \frac{1}{4}, \\
&P(ABC) = P(\{HH\}) = 1/4.
\end{aligned}
$$

Therefore,

$$P(AB) = P(A)P(B), \quad P(AC) = P(A)P(C),$$

$$P(BC) = P(B)P(C).$$

However,

$$P(ABC) \neq P(A)P(B)P(C).$$

So A, B and C are pairwise independent. $\square$

Independence of Several Events

Example

Toss two different standard dice, white and black. Define the following events:
$A_1 = \{\text{first die} = 1, 2 \text{ or } 3\}$;
$A_2 = \{\text{first die} = 3, 4 \text{ or } 5\}$;
$A_3 = \{\text{sum of faces is } 9\}$.
Are events A_1, A_2 and A_3 mutually independent?

Solution. The sample space Ω of the outcomes consists of all ordered pairs $(i, j), i, j = 1, \cdots, 6$, i.e.,
$\Omega = \{(1, 1), (1, 2), ..., (6, 6)\}$. Since the two dice are standard, each point in S has probability $1/36$. So the probability of the events are

$$P(A_1) = 1/2, \quad P(A_2) = 1/2, \quad P(A_3) = 1/9.$$

Independence of Several Events

Solution. Obviously,

$$\begin{aligned} &A_1A_2 = \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\}, \\ &A_1A_3 = \{(3,6)\},\ A_2A_3 = \{(3,6),(4,5),(5,4)\}, \\ &A_1A_2A_3 = \{(3,6)\}. \end{aligned}$$

It follows that

$$P(A_1A_2A_3) = 1/36 = (1/2)(1/2)(1/9) = P(A_1)P(A_2)P(A_3).$$

However,

$$\begin{aligned} P(A_1A_2) &= 1/6 \neq 1/4 = P(A_1)P(A_2), \\ P(A_1A_3) &= 1/36 \neq 1/18 = P(A_1)P(A_3), \\ P(A_2A_3) &= 1/12 \neq 1/18 = P(A_2)P(A_3). \end{aligned}$$

So events A_1, A_2 and A_3 are not mutually independent. $\square$

Independence of Several Events

Definition

Events $A_1, A_2, \cdots, A_n$ *are* ***pairwise independent*** *if every possible pair of these events are independent, i.e.,* $P(A_j A_k) = P(A_j) \cdot P(A_k)$ *for all* $j, k (j \neq k)$.

Definition

The n *events* $A_1, A_2, \cdots, A_n$ *are* ***independent*** *if, for every subset* $A_{i_1}, A_{i_2}, \cdots, A_{i_k}$ *of these events* $(k = 2, 3, \cdots, n)$,

$$P(A_{i_1} A_{i_2} \cdots A_{i_k}) = P(A_{i_1}) P(A_{i_2}) \cdots P(A_{i_k}). \tag{2}$$

Independence of Several Events

Theorem

The n events $A_1, A_2, \cdots, A_n$ are ***independent*** *if and only if for any subset $A_{i_1}, A_{i_2}, \cdots, A_{i_k}$,*

$$P(A_{i_1}^* A_{i_2}^* \cdots A_{i_k}^*) = P(A_{i_1}^*)P(A_{i_2}^*) \cdots P(A_{i_k}^*), \quad (k = 2, 3, \cdots, n) \tag{3}$$

where A_i^* denotes either A_i or $\overline{A}_i$ (the same on both sides of the equation).

Independence of Several Events

Example

Suppose that A, B and C are three independent events such that $P(A) = 1/4, P(B) = 1/3$ and $P(C) = 1/2$. (a) Determine the probability that none of these three events will occur; (b) Determine the probability that exact one of these three events will occur.

Solution. (a) Since A, B and C are three independent events,

$$P(\overline{A}\,\overline{B}\,\overline{C}) = P(\overline{A})P(\overline{B})P(\overline{C}) = (1-1/4)(1-1/3)(1-1/2) = 1/4.$$

(b) By additivity of probability and independency,

$$\begin{aligned} & P(A\overline{B}\,\overline{C} \cup \overline{A}B\overline{C} \cup \overline{A}\,\overline{B}C) \\ = {} & P(A\overline{B}\,\overline{C}) + P(\overline{A}B\overline{C}) + P(\overline{A}\,\overline{B}C) \\ = {} & P(A)P(\overline{B})P(\overline{C}) + P(\overline{A})P(B)P(\overline{C}) + P(\overline{A})P(\overline{B})P(C) \end{aligned}$$

$\square$

Solution. (b) By additivity of probability and independency,

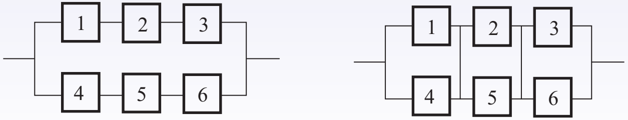
$$
\begin{aligned}
& P(A\overline{B}\,\overline{C} \cup \overline{A}B\overline{C} \cup \overline{A}\,\overline{B}C) \\
= & P(A\overline{B}\,\overline{C}) + P(\overline{A}B\overline{C}) + P(\overline{A}\,\overline{B}C) \\
= & P(A)P(\overline{B})P(\overline{C}) + P(\overline{A})P(B)P(\overline{C}) + P(\overline{A})P(\overline{B})P(C) \\
= & (1/4)(1-1/3)(1-1/2) + (1-1/4)(1/3)(1-1/2) \\
& + (1-1/4)(1-1/3)(1/2) \\
= & 1/12 + 1/8 + 1/4 = 11/24.
\end{aligned}
$$

$\square$

Independence of Several Events

Example

(**Reliability analysis**) We have two systems illustrated in following Figure. Given a particular lifetime value t_0, let A_i denote the event that the lifetime of cell i exceeds $t_0 (i = 1, 2, \cdots, 6)$. Assume that the $A_i s$ are independent events (whether any particular cell lasts more than t_0 hours has no bearing on whether or not any other cell does) and that $P(A_i) = 0.9$ for every i since the cells are identical. Determine the probabilities that the systems' lifetime exceeds t_0. Which system is more reliable?



(a) (b)

Independence of Several Events

Solution. For the first system,

$$
\begin{aligned}
& P(\text{system lifetime exceeds } t_0) \\
= & P\big((A_1 \cap A_2 \cap A_3) \cup (A_4 \cap A_5 \cap A_6)\big) \\
= & P(A_1 \cap A_2 \cap A_3) + P(A_4 \cap A_5 \cap A_6) - \\
& P\big((A_1 \cap A_2 \cap A_3) \cap (A_4 \cap A_5 \cap A_6)\big) \\
= & P(A_1)P(A_2)P(A_3) + P(A_4)P(A_5)P(A_6) - \\
& P(A_1)P(A_2)P(A_3)P(A_4)P(A_5)P(A_6) \\
= & 0.9^3 + 0.9^3 - 0.9^6 \approx 0.927.
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
& P(\text{system lifetime exceeds } t_0) \\
= & 1 - P(\text{both subsystem lives are } \leqslant t_0) \\
= & 1 - [P(\text{subsystem lives are } \leqslant t_0)]^2 \\
= & 1 - [1 - P(\text{subsystem lives are } > t_0)]^2 \\
= & 1 - [1 - (0.9)^3]^2 = 0.927.
\end{aligned}
$$

Independence of Several Events

Solution. For the second configuration,

$$
\begin{aligned}
&P(\text{system lifetime exceeds } t_0)\\
&= P\big((A_1 \cup A_4) \cap (A_2 \cup A_5) \cap (A_3 \cup A_6)\big)\\
&= P(A_1 \cup A_4) \cdot P(A_1 \cup A_4) \cdot P(A_1 \cup A_4)\\
&= \big[P(A_1) + P(A_4) - P(A_1 A_4)\big]^3\\
&= (0.9 + 0.9 - 0.9 \cdot 0.9)^3 \approx 0.97
\end{aligned}
$$

In reliability theory, we call the probability $P(\text{system lifetime exceeds } t_0)$ the reliability of the system. Since the reliability of the second system is larger than the first, the second system is more reliable. □

Contents

Bernoulli Trials

Consider the following experiments:

- Flipping a coin and observing which side will appears.
- Rolling a die and observing whether the number six appears.
- In conducting a political opinion poll, choosing a voter at random to ascertain whether that voter will vote "yes" in an upcoming referendum.

All above examples have the same character: the experiment E that has *only two outcomes*. A ***Bernoulli experiment*** E is such kind of random experiment, the outcome of which can be classified in but one of two mutually exclusive and exhaustive ways, mainly, *success* (denoted by S) or *failure* (denoted by F) (e.g., female or male, life or death, nondefective or defective).

Bernoulli Trials

A sequence of Bernoulli trials E_n occurs when a Bernoulli experiment is *performed several independent times* so that *the probability of success*, say, *p, remains the same* from trial to trial. That is, in such a sequence we let p denote the probability of success on each trial. In addition, frequently $q = 1 - p$ denote the probability of failure; that is, we shall use q and $1 - p$ interchangeably.

Bernoulli Trials

Theorem

The probability that the outcome of an experiment that consists of n Bernoulli trials has k successes and $n-k$ failures is given by:

$$P_n(k) := P_n(\#S = k) = \binom{n}{k} p^k q^{n-k}.$$

Here $\#S$ means the number of successes.

Proof. Now, suppose that, when we performed the n trials, we first obtained k consecutive successes S and then $n-k$ consecutive failures F. By independence,

$$P(\underbrace{SS\cdots S}_{k\ times}\ \underbrace{FF\cdots F}_{(n-k)\ times}) = P(S)^k P(F)^{n-k} = p^k(1-p)^{n-k}.$$

Bernoulli Trials

Hence, given that we can place the k successes among the n trials in $\binom{n}{k}$ different ways, we deduce that

$$P_n(\#S = k) = \binom{n}{k} p^k q^{n-k}.$$

$\square$

Summary

- A and B are independent if and only if $P(AB) = P(A)P(B)$.
- The n events $A_1, A_2, \cdots, A_n$ are **independent** if and only if for any subset $A_{i_1}, A_{i_2}, \cdots, A_{i_k}$,

  $$P(A^*_{i_1} A^*_{i_2} \cdots A^*_{i_k}) = P(A^*_{i_1}) P(A^*_{i_2}) \cdots P(A^*_{i_k}), \quad (k = 2, 3, \cdots, n)$$

  where A^*_i denotes either A_i or $\overline{A}_i$ (the same on both sides of the equation).

Thank you for your patience !