Section 3.2 Independence of Random Variables

School of Sciences, BUPT

Independence of Random Variables

$$F(x, y) \longrightarrow F_X(x),\ F_Y(y)$$

$$F_X(x),\ F_Y(y) \longrightarrow F(x, y)?$$

Generally, NO!

Independence of Random Variables

In Chapter 2, we pointed out that one way of defining independence of two events A and B is via the condition $P(A \cap B) = P(A) \cdot P(B)$. Here is an analogous definition for the independence of two random variables. Intuitively, if and only if for any subsets I_1, $I_2 \in \mathbb{R}$,

$$P(X \in I_1, Y \in I_2) = P(X \in I_1)P(Y \in I_2) \tag{1}$$

holds, we may obtain the independence of X and Y. However,

Independence of Random Variables

Definition

Two ***random variables*** X ***and*** Y ***are said to be independent*** *if for every pair of* x *and* y,

$$F(x,y) = F_X(x)F_Y(y). \tag{2}$$

We denote it by $X \perp Y$.

Independence of Random Variables

In fact, the following definition of independence of two random variables is equivalent to Definition 1, which is more useful in many applications.

Definition

*Two **random variables** X **and** Y **are said to be independent** if for every pair of x and y,*

$$\begin{cases} p(x,y) = p_X(x) \cdot p_Y(y) & \textit{when } X \textit{ and } Y \textit{ are discrete} \\ \quad\quad\textit{or} & \\ f(x,y) = f_X(x) \cdot f_Y(y) & \textit{when } X \textit{ and } Y \textit{ are continuous.} \end{cases} \tag{3}$$

If (3) is not satisfied for all (x,y), then X and Y are said to be dependent.

Independence of Random Variables

Example

Suppose that $X \perp Y$.

x	-2	-1	0	$1/2$
$p_X(x)$	$1/4$	$1/3$	$1/12$	$1/3$

y	$-1/2$	1	3
$p_Y(y)$	$1/2$	$1/4$	$1/4$

Find the joint p.f. of (X, Y).

Independence of Random Variables

Solution. Since $X \perp Y$,

$$P(X=-2, Y=-1/2) = P(X=-2)P(Y=-1/2) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}.$$

Similarly, we can obtain the joint probability table of X and Y in Table

y \ x	-2	-1	0	$1/2$	$p_Y(y)$
$-1/2$	$1/8$	$1/6$	$1/24$	$1/6$	$1/2$
1	$1/16$	$1/12$	$1/48$	$1/12$	$1/4$
3	$1/16$	$1/12$	$1/48$	$1/12$	$1/4$
$p_X(x)$	$1/4$	$1/3$	$1/12$	$1/3$	1

□

Independence of Random Variables

Example

Suppose that the joint probability table of X and Y is in Table.

$y \backslash x$	1	2	3
1	$1/6$	$1/9$	$1/18$
2	$1/3$	$1/a$	$1/b$

Find the values of a and b such that $X \perp Y$.

Independence of Random Variables

Solution. For every pair of x and y, we need

$$p(x, y) = p_X(x) \cdot p_Y(y).$$

Select two simple equalities

$$p(2, 1) = p_X(2)p_Y(1) \text{ and } p(3, 1) = p_X(3)p_Y(1).$$

That is,

$$\frac{1}{9} = \frac{1}{3}\Big(\frac{1}{9} + \frac{1}{a}\Big), \quad \frac{1}{18} = \frac{1}{3}\Big(\frac{1}{18} + \frac{1}{b}\Big).$$

Thus we get $a = 9/2, b = 9$. □

Independence of Random Variables

Example

Suppose that n machines produce components, and all the machines produce the same number c of components. We pick one at random and let its serial number be (X, Y), where X is the machine number and Y is the component index. Are X and Y independent?

Solution. Since the joint p.d.f. of (X, Y) is

$$f(x, y) = (nc)^{-1}, \quad 1 \leqslant x \leqslant n,\ 1 \leqslant y \leqslant c$$

and

$$f_X(x) = \frac{1}{n},\ 1 \leqslant x \leqslant n, \ \text{ and } f_Y(y) = \frac{1}{c},\ 1 \leqslant y \leqslant c.$$

Obviously, $f(x, y) = f_X(x) f_Y(y)$ for all (x, y). So X and Y are independent in this case. □

Independence of Random Variables

Example

Suppose that $(X, Y) \sim N(\mu_1, \mu_2; \sigma_1^2, \sigma_2^2; \rho)$. Prove $X \perp Y$ if and only if $\rho = 0$.

Solution. If $\rho = 0$, then the independence of X and Y is obvious. Conversely, since $f(x, y) = f_X(x) f_Y(y)$ for any x and y,

$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(x-\mu_2)^2}{\sigma_2^2}\right]}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1}} \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y-\mu_2)^2}{2\sigma_2}}.$$

Because x and y are arbitrary, we take $x = \mu_1$, $y = \mu_2$. Then

$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} = \frac{1}{\sqrt{2\pi}\sigma_1}\frac{1}{\sqrt{2\pi}\sigma_2},$$

i.e., $\rho = 0$. □

Independence of Random Variables

Example

Are X and Y independent?
(a) Suppose that (X, Y) has a uniform distribution over $D = \{(x, y) : 0 < x < a, 0 < y < b\}$.
(b) Suppose that (X, Y) has a uniform distribution over the unit circle.

Solution. (a) By using the result in Example **??**, we know for any $x, y \in D$, $f(x, y) = f_X(x) f_Y(y)$. Thus X and Y are independent.

Independence of Random Variables

(b) The joint p.d.f. of (X, Y) is

$$f(x,y) = \begin{cases} \dfrac{1}{\pi} & \text{for } x^2 + y^2 < 1, \\ 0 & \text{otherwise.} \end{cases}$$

The marginal p.d.f. of X and Y are

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2}{\pi}\sqrt{1-x^2}, \quad |x| < 1,$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)dx = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi} dx = \frac{2}{\pi}\sqrt{1-y^2}, \quad |y| < 1.$$

Since $f(x,y) \neq f_X(x) f_Y(y)$, X and Y are dependent. □

Independence of Random Variables

In fact, we have a necessary and sufficient condition for the random variables X and Y to be independent in a rectangular.

Theorem

Let a, b, c and d be given values such that $-\infty \leqslant a < b \leqslant +\infty$ and $-\infty \leqslant c < d \leqslant +\infty$, and let S be a rectangle in the xy-plane:

$$S = \{(x, y) \; |a \leqslant x \leqslant b \text{ and } c \leqslant y \leqslant d\}.$$

Suppose that $f(x, y) = 0$ for every point (x, y) outside S. Then the continuous (discrete) random variables X and Y are independent if and only if their joint p.d.f. (or p.f.) can be expressed as

$$f(x, y) = h(x)g(y)$$

for all points in S.

Independence of Random Variables

Example

Suppose that the joint p.d.f. of X and Y is

$$f(x, y) = 6e^{-2x}e^{-3y}, \quad 0 < x < \infty,\ 0 < y < \infty$$

and is equal to 0 outside this region. Are the random variables independent?

Solution. The joint density function factors, and thus the random variables, are independent (with one being exponential with rate 2 and the other exponential with rate 3). $\square$

Independence of Random Variables

If the joint density function is

$$f(x, y) = 24xy, \quad 0 < x < 1,\ 0 < y < 1,\ 0 < x + y < 1$$

and is equal to 0 otherwise, are the random variables independent?

Theorem

Let X and Y be two independent random variables. Then for any real functions $g(\cdot)$ and $h(\cdot)$, $g(X)$ and $h(Y)$ are independent.

Summary

Theorem

*Two **random variables** X **and** Y **are independent** if and only if for every pair of x and y,*

$$\begin{cases} p(x,y) = p_X(x) \cdot p_Y(y) & \textit{when } X \textit{ and } Y \textit{ are discrete} \\ \qquad\quad \textit{or} & \\ f(x,y) = f_X(x) \cdot f_Y(y) & \textit{when } X \textit{ and } Y \textit{ are continuous.} \end{cases} \tag{4}$$

Theorem

Let X and Y be two independent random variables. Then for any real functions $g(\cdot)$ and $h(\cdot)$, $g(X)$ and $h(Y)$ are independent.

Thank you for your patience !